HEAVY TAILS AND NEGATIVE CORRELATION IN A BINOMIAL MODEL FOR SPORTS MATCHES: APPLICATIONS TO CURLING

JOHN FRY, MARK AUSTIN, AND SILVIO FANZON*

Abstract. A binomial model for sports matches is developed making use of the maximum possible score n in a game. In contrast to previous approaches the scores of the two teams are negatively correlated, abstracting from a scenario whereby teams cancel each other out. When n is known, analytical results are possible via a Gaussian approximation. Model calibration is obtained via generalized linear modelling, enabling elementary econometric and strategic analysis to be performed. Inter alia this includes quantifying the Last Stone First End effect, analogous to the home-field advantage found in conventional sports. When n is unknown the model behaviour is richer and leads to heavy-tailed non-Gaussian behaviour. We present an approximate analysis of this case based on the Variance Gamma distribution.

Keywords: Binomial Distribution; Curling; Normal Mean Variance Mixture; Sports; Variance Gamma distribution.

2020 Mathematics Subject Classification: 62P25, 91-10, 62M99.

1. Introduction

There is enduring interest in binomial and Poisson modelling of sporting contests (Baker et al., 2022; Singh et al., 2023; Scarf et al., 2022). There has also been significant prior academic attention paid to curling, although primarily from applied mathematical (Lawson and Rave, 2020) and strategic (Willoughby and Kostuk, 2004, 2005) perspectives. In contrast our paper offers a more foundational probabilistic approach. Specifically, based on the characteristics of curling matches (see Section 2), we develop a binomial model for curling. This approach may be of wider theoretical interest in generating sports models with negative correlations (Malinovsky and Rinott, 2023). For the curling application our approach enables novel empirical analyses to be performed, see Section 4.

From a theoretical perspective the importance of our approach is three-fold. Firstly, our model is parameterised in terms of the maximum possible score n and the probability of scoring in each of the trials. The result is that the score for each team is negatively correlated, resulting from teams cancelling each other out. This contrasts with much of the prevailing literature in which the scores of each team are assumed to be either independent or positively correlated (Scarf et al., 2022). Secondly, if n is known, analytical results are possible based on a Gaussian approximation (Scarf et al., 2019). The result is shown to have potential relevance for betting markets. Thirdly, if n is unknown and varies statistically, this can lead to heavy-tailed non-Gaussian behaviour via a normal mean-variance mixture construction (Bingham and Kiesel, 2001). Here, we present an approximate analysis based on the Variance Gamma distribution (Finlay and Seneta, 2006).

From an applications perspective the structure of our model allows for parameters to be calibrated to historical results via generalised linear modelling (see Section 4). This enables us to measure teams' offensive and defensive capabilities and sheds new light on the interplay between offensive and defensive strategies in elite-level curling. The model also enables us to quantify the effects of the Last Stone First End advantage. The latter is analogous to the home-field advantage seen elsewhere (Boudreaux et al., 2017; Ehrlich and Potter, 2023).

The layout of this paper is as follows. An overview of the game of curling is given in Section 2. The proposed statistical model is outlined in Section 3. An empirical application is discussed in Section 4. Section 5 concludes and discusses the opportunities for further research. An Appendix containing proofs to mathematical statements is included at the end of the paper.

(John Fry, Mark Austin, Silvio Fanzon) Department of Mathematics, University of Hull, Hull, HU6 7RX, UK
E-mail addresses: J.M.Fry@hull.ac.uk, mdaustin2k@yahoo.co.uk.
*Corresponding author: S. Fanzon (S.Fanzon@hull.ac.uk).

2. Overview of the game of curling

Curling is a sport that is played on ice between two teams. The teams take it in turns to slide stones towards a target, known as the House. Traditional teams are made up of four players, either all men or all women. The four players are referred to as Lead, Second, Third and Fourth (usually the position occupied by the Skip). It is worth mentioning that a mixed event is present at Olympic level which is only played as mixed doubles, i.e. one male and one female. This event has slightly different rules to that of the four person teams.

During international competitions a game of curling is played over ten ends, where an end is made up of sixteen stones: eight delivered by each team. If the two teams remain level after ten ends, the teams play to sudden death, meaning that the next score wins. The eight stones, two by each player, are delivered with the aim of outscoring the opposition. The team that has the last stone in an end has what is called the hammer. The score at the end of each end is calculated by the number of consecutive stones that are in the house that are closer to the centre, known as the button, than any of the opposition's stone. If no stones are in the house at the end of the end, then no points are scored, resulting this in a blank end. If a team scores points in an end, then the hammer passes onto the opposition. Instead if the end is blank, the team that has the hammer retains the hammer. The advantage of having the hammer in the first end is known as Last Stone First End (LSFE) and is decided by a last stone draw. This is a trial of skill at the start of the match, markedly different from e.g. the coin toss in soccer, and entails two players from each team throwing one stone each towards the house. The combined distance from the button is calculated and the team with the lowest combined distance from the button will get the LSFE advantage. This LSFE is equivalent to the home-field advantage in conventional sports (Boudreaux et al., 2017; Ehrlich and Potter, 2023) and can be quantified using a generalised linear modelling approach in Section 4.

3. The model

The proposed model is based on the outline of a curling match presented in Section 2, and builds on a related model in (Baker et al., 2022). The model construction is of theoretical interest in providing a physically realistic mechanism, teams cancelling each-other out, that can generate both negative correlations and heavy tails. The model is also amenable to relatively straightforward calibration to historical data, see Section 4.

Suppose that a match between two teams Team X and Team Y consists of a sequence of n iid trials. The parameter n can be either known or unknown. In (Baker et al., 2022) the parameter n has a Poisson distribution and each trial has only 2 possible outcomes. In contrast, for our statistical sports model, each trial has 3 possible outcomes. Specifically, in each trial X scores 1 point with probability p_X, Y scores 1 point with probability p_Y, and 0 points are scored with probability $1 - p_X - p_Y$. This latter possibility gives an explicit characterisation of defensive play whereby teams cancel each other out. The classical model of independent scores (Scarf et al., 2019) is recovered in the special case $p_Y = 1 - p_X$.

3.1. The case of known n.

In this section we suppose that the number of iid trials n is given. Let X and Y be the number of points scored by Teams X and Y, respectively. The total number of points scored by both teams in the match is then given by $X + Y$. We have that X, Y and $X + Y$ follow binomial distributions, with

$$X \sim \mathrm{Bin}(n, p_X);\ Y \sim \mathrm{Bin}(n, p_Y);\ X + Y \sim \mathrm{Bin}(n, p_X + p_Y). \tag{3.1}$$

From equation (3.1) we infer that

$$\begin{aligned} \mathrm{Var}[X+Y] &= n(p_X + p_Y)(1 - p_X - p_Y) \\ &= np_X(1-p_X) + np_Y(1-p_Y) - 2np_Xp_Y \\ &= \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\mathrm{Cov}(X,Y)\,. \end{aligned} \tag{3.2}$$

Equations at 3.2 allow to conclude that X and Y are negatively correlated, with

$$\mathrm{Cov}(X,Y) = -np_Xp_Y\,. \tag{3.3}$$

An approximate formula for match outcomes can now be calculated using a Gaussian approximation (Scarf et al., 2019), as outlined in Proposition 1. This result has its own theoretical interest, as well as some relevance for potential sports-betting applications (Fry et al., 2021). The proof is deferred to Appendix A.1.

Proposition 1 (Approximate probabilities of match outcomes)**.** *Consider the binomial model outlined above. Denoting by* Φ *the CDF of the standard normal distribution, we have:*

(1) The probability of Team X winning is approximately equal to

$$Pr(X \text{ wins}) \approx \Phi\left(\frac{np_X - np_Y - 0.5}{\sqrt{np_X(1-p_X) + np_Y(1-p_Y) + 2np_Xp_Y}}\right).$$

(2) The probability of a draw is approximately equal to

$$\begin{aligned} Pr(Draw) &\approx \Phi\left(\frac{0.5 - np_X + np_Y}{\sqrt{np_X(1-p_X) + np_Y(1-p_Y) + 2np_Xp_Y}}\right) \\ &+ \Phi\left(\frac{0.5 + np_X - np_Y}{\sqrt{np_X(1-p_X) + np_Y(1-p_Y) + 2np_Xp_Y}}\right) - 1. \end{aligned}$$

An approximate formula for curling match outcomes can now be calculated by replacing $n = 80$ in the formulas of Proposition 1, and by making minor amendments given the nature of curling matches as outlined in Section 2. The proof is found in Appendix A.2.

Proposition 2 (Approximate probabilities of curling-match outcomes)**.** *Consider the binomial model outlined above and the nature of curling matches as outlined in Section 2. Then:*

(1) The probability of Team X winning after 10 *ends is approximately equal to*

$$Pr(X \text{ wins after 10 ends}) \approx \Phi\left(\frac{80p_X - 80p_Y - 0.5}{\sqrt{80p_X(1-p_X) + 80p_Y(1-p_Y) + 160p_Xp_Y}}\right).$$

(2) The probability of a draw at the end of 10 ends is approximately equal to

$$\begin{aligned} Pr(Draw) &\approx \Phi\left(\frac{0.5 - 80p_X + 80p_Y}{\sqrt{80p_X(1-p_X) + 80p_Y(1-p_Y) + 160p_Xp_Y}}\right) \\ &+ \Phi\left(\frac{0.5 + 80p_X - 80p_Y}{\sqrt{80p_X(1-p_X) + 80p_Y(1-p_Y) + 160p_Xp_Y}}\right) - 1. \end{aligned}$$

(3) The probability that Team X wins overall is approximately equal to

$$Pr(X \text{ wins}) \approx Pr(X \text{ wins after 10 ends}) + Pr(Draw)\left(\frac{p_X}{p_X + p_Y}\right).$$

3.2. **The case of unknown** n**.** Suppose that the number of iid trials n is unknown. It is natural to assume that n varies statistically, and that an approximately conditional Gaussian structure is retained for the distribution of the points difference $X - Y$. This is in line with the results in Proposition 1 above. Using equations (3.2)-(3.3), the approximate distribution of the points difference between the two teams is then given by

$$X - Y \,|\, n \overset{d}{\approx} N\left(n(p_X - p_Y), n(p_X(1-p_X) + p_Y(1-p_Y) + 2p_Xp_Y)\right). \tag{3.4}$$

The points difference between the two teams can be constructed according to the following sequence of steps

(1) Generate n from a given distribution on $[0, \infty)$.
(2) $X - Y|n$ is distributed according to (3.4)

In the notation of (Bingham and Kiesel, 2001) equation (3.4) corresponds to a normal mean-variance mixture model with parameter $\mu = 0$. A reasonable modelling assumption is to assume that n is gamma distributed with parameters

$$n \sim \Gamma\left(\lambda, \frac{\lambda}{\hat{n}}\right). \tag{3.5}$$

Equation (3.5) is a continuous approximation which retains the positivity of n and allows a large number of different distributional shapes. Moreover, $\hat{n}$ is the expected value of n, whilst the parameter λ is related to a scale parameter representing uncertainty around the expected value (Bingham and Fry, 2010). This construction leads to an approximate analysis based on the Variance Gamma distribution, see Definition 1 below. This approach is interesting because it develops a principled way of introducing non-Gaussian behaviour into observable scoring patterns. Consideration of the Variance Gamma distribution also builds on a number of pertinent recent statistical applications (Fischer et al., 2024) including sports (Fry et al., 2021) and finance (**?**).

Definition 1 (Variance Gamma distribution)**.** *The Variance Gamma distribution with parameters $\alpha, \beta, \lambda, \mu$ is the continuous random variable defined on $(-\infty, \infty)$ with probability density function*

$$f(x) := \frac{\gamma^{2\lambda}}{\sqrt{\pi}\,\Gamma(\lambda)} \left(\frac{|x-\mu|}{2\alpha}\right)^{\lambda-\frac{1}{2}} e^{\beta(x-\mu)}\ K_{\lambda-\frac{1}{2}}(\alpha|x-\mu|)\,, \quad \gamma := \sqrt{\alpha^2-\beta^2}\,,$$

where $\Gamma(\cdot)$ is the Gamma function, and $K_\rho(\cdot)$ is the modified Bessel function of the third kind. The CDF of a Variance Gamma distribution with parameters $\alpha, \beta, \lambda, \mu$ is denoted by $\mathrm{VG}_{\alpha,\beta,\lambda,\mu}(x)$.

The following proposition shows that the Variance Gamma distribution has an explicit representation in terms of a normal mean-variance mixture model. The proof is deferred to Appendix A.3.

Proposition 3 (Variance gamma representation theorem)**.** *Suppose given the following normal mean variance mixture representation:*
 (1) Generate $n \sim \Gamma\left(\lambda, \frac{\lambda}{\hat{n}}\right)$.
 (2) $X|n \sim N(n\mu, n\sigma^2)$.

Then the distribution of X is variance gamma with parameters

$$\alpha = \sqrt{\frac{2\lambda}{\hat{n}\sigma^2} + \frac{\mu^2}{\sigma^4}}\,, \quad \beta = \frac{\mu}{\sigma^2}\,, \quad \lambda = \lambda\,, \quad \mu = 0\,, \quad \gamma = \sqrt{\alpha^2-\beta^2} = \sqrt{\frac{2\lambda}{\hat{n}\sigma^2}}$$

Combining the normal mean-variance mixture representation in (3.4) with the gamma distribution in equation (3.5) leads to what we term the Variance Gamma match:
 (1) Generate $n \sim \Gamma\left(\lambda, \frac{\lambda}{\hat{n}}\right)$.
 (2) $X - Y|n \sim N(n(p_X - p_Y), n(p_X(1-p_X) + p_Y(1-p_Y) + 2p_Xp_Y))$.

Proposition 4 outlines match outcome probabilities and potential sports betting opportunities for the Variance Gamma match. A proof is given in Appendix A.4.

Proposition 4 (Match outcomes in the Variance Gamma match)**.** *Consider the Variance Gamma match outlined above and define parameters*

$$\alpha := \sqrt{\frac{2\lambda}{\hat{n}\sigma^2} + \frac{\mu^2}{\sigma^4}}, \qquad \beta := \frac{\mu}{\sigma^2}\,, \qquad \gamma := \sqrt{\alpha^2-\beta^2}\,,$$
$$\mu := p_X - p_Y\,, \qquad \sigma^2 := p_X(1-p_X) + p_Y(1-p_Y) + 2p_Xp_Y\,.$$

We have the following approximation formulae:
 (1) The probability of Team X winning is approximately equal to

$$Pr(X\ wins) \quad \approx \quad \mathrm{VG}_{\alpha,\beta,\lambda,0}(0.5)\,.$$

 (2) The probability of a draw is approximately equal to

$$Pr(Draw) \quad \approx \quad \mathrm{VG}_{\alpha,\beta,\lambda,0}(0.5) - \mathrm{VG}_{\alpha,\beta,\lambda,0}(-0.5)\,.$$

Next, consider when the variance gamma approximation may be warranted. Commonly published statistical tables for the t-distribution, see e.g. those in (Fry and Burke, 2022), typically go up to around 120 degrees of freedom. This suggests that at this point the excess kurtosis associated with the t_{121} distribution ceases to be a practical problem. We have the following result:

$$\text{If } X \sim t_r;\ \text{Kurtosis}[X] = \frac{3r-6}{r-4}. \tag{3.6}$$

Plugging in $r = 121$ to equation (3.6) gives kurtosis of 3.05. This suggests that excess kurtosis should cease being a practical problem once the kurtosis lies below 3.05, leading to the following proposition:

Proposition 5 (Practical implementation)**.** *The Variance Gamma approximation may be needed once the kurtosis of the distribution of points differences exceeds 3.05.*

4. Empirical Application

In this section we estimate the model presented in Section 3 with an application to historical results for a sequence of 583 men's international curling matches from 2019 to 2023. The dataset is available from the authors upon request. Following a similar approach in (Fry et al., 2021) model parameters are estimated via generalised linear models. Dummy variables for each team are included that abstract from teams' offensive strengths. If the coefficient of the team parameter is positive (negative) this suggests the team has greater (less) than average offensive skill. Dummy variables for each opponent are included that abstract from teams' defensive strengths. If the coefficient of the opponent parameter is negative (positive) this suggests the team has greater (less) than average defensive skill. Also included in the model is a dummy variable corresponding to the LSFE. This corresponds to a small advantage associated with the team that starts second, analogous to the home-field advantage found in other sports (Boudreaux et al., 2017; Ehrlich and Potter, 2023), and determined by a trial of skill called the "draw-to-button shootout" at the start of the match.

The above set up leads to a deceptively complex logistic regression problem to determine the probabilities p_X and p_Y for Team X and Team Y to score a single point, respectively. These are characterised by 1166 observations with 44 variables. Here, this complexity is resolved by stepwise regression (Fry and Burke, 2022). Results for the final model chosen are shown in Table 1. Alternative specifications based on probit and Poisson generalized linear models (not reported) yield similar results.

Coefficient	**Estimate**	**E.S.E.**	**t-value**	**p-value**
(Intercept)	-2.5033	0.0286	-87.6780	0.0000
LSFE	0.1268	0.0257	4.9250	0.0000
Opponent = Sweden	-0.4832	0.0538	-8.9790	0.0000
Opponent = Scotland	-0.3853	0.0512	-7.5250	0.0000
Opponent = Canada	-0.3313	0.0609	-5.4400	0.0000
Opponent = Italy	-0.2559	0.0480	-5.3340	0.0000
Opponent = Switzerland	-0.2574	0.0484	-5.3220	0.0000
Team = Sweden	0.2870	0.0451	6.3630	0.0000
Team = Canada	0.3005	0.0540	5.5610	0.0000
Team = Scotland	0.2484	0.0456	5.4490	0.0000
Team = Italy	0.2197	0.0462	4.7610	0.0000
Team = Switzerland	0.1962	0.0465	4.2230	0.0000
Team = USA	0.2085	0.0560	3.7240	0.0002
Team = Norway	0.1175	0.0478	2.4550	0.0141
Opponent = USA	-0.1903	0.0574	-3.3150	0.0009
Opponent = Norway	-0.1508	0.0460	-3.2750	0.0011
Team = Newzealand	-0.4079	0.1560	-2.6140	0.0089
Team = Poland	-0.3734	0.1773	-2.1060	0.0352
Team = China	-0.1492	0.0805	-1.8520	0.0640
Opponent = Japan	-0.1212	0.0608	-1.9930	0.0462
Team = Russia	0.1027	0.0610	1.6830	0.0924
Team = Finland	-0.1432	0.0888	-1.6130	0.1068

Table 1. Generalised linear model results applied to historical international curling matches.

From the order in which variables appear in Table 1 result suggest that it is the LSFE variable that discriminates most between the teams involved. Since the LSFE results from a skills-based draw-shot challenge at the start of the match, this variable arguably gives a measure of intrinsic curling aiming ability in idealised settings. After that, it is the teams' level of defensive ability that provides the largest amount of discrimination. The importance of this observation is twofold. Firstly, it appears that curling is a primarily defensive sport. Secondly, results also tally with previous suggestions from sports analytics that place a surprisingly high value upon good defensive play (McHale et al., 2012). A curling-based interpretation of these results based on bond-rating terminology is given in Table 2. In Table 2 Japan's rating above Russia reflects the apparent premium placed on defensive skills within curling. Table 3 illustrates calculations associated with Proposition 2 given the regression output in Table 1.

Team	**Rating**	**Interpretation**
Canada, Italy, Norway Scotland, Sweden, Switzerland USA	AAA	Above average attack, above average defence
Japan	AA+	Above average defence, average attack
Russia	AA	Above average attack, average defence
Czech Republic, Denmark, England Germany, Korea, Netherlands Spain, Turkey	AA–	Average attack, average defence
China, Finland New Zealand, Poland	A+	Average attack, below average defence

Table 2. Suggested curling-based interpretation of generalised linear model output.

Suppose Sweden plan Canada and Sweden have the LSFE.
Using Team = Sweden, Opponent = Canada gives
$\text{logit}(p_S) = -2.5033 + 0.1268 + 0.2870 - 0.3313 = -2.4208;\ p_S = 0.08160028.$
Similarly, using Team = Canada, Opponent = Sweden gives
$\text{logit}(p_C) = -2.5033 + 0.3005 - 0.4832 = -2.686;\ p_C = 0.06380454.$
Inputting these numbers into formulas in Proposition 3 then gives
$Pr(\text{Sweden wins after 10 ends}) = \Phi\left(0.9236592/3.406912\right) = 0.6068481.$
$Pr(\text{Draw}) = \Phi\left(-0.9236592/3.406912\right) + \Phi\left(1.923659/3.406912\right) - 1$
$Pr(\text{Draw}) = 0.3931519 + 0.7138387 - 1 = 0.1069907$
$Pr(\text{Sweden wins}) = 0.6068481 + 0.1069907\left(\frac{P_S}{P_S+P_C}\right) = 0.6668906.$
If instead Canada have the LSFE the above switches to
$\text{logit}(p_S) = -2.5033 + 0.2870 - 0.3313 = -2.5476;\ p_S = 0.07258789$
Similarly, using Team = Canada, Opponent = Sweden gives
$\text{logit}(p_C) = -2.5033 + 0.1268 + 0.3005 - 0.4832 = -2.5592;\ p_C = 0.07181085.$

Table 3. Example calculations based on Proposition 2 using generalised linear model output.

5. Conclusions and further work

Inspired by applications to curling we develop a binomial model for sports matches. Our paper reflects much recent interest in sports (Baker et al., 2022; Fry et al., 2024; Singh et al., 2023). Results are interesting in terms of the empirical analysis of curling, as well as being suggestive of potential betting-market applications. In addition, results obtained are also of wider theoretical interest. Indeed, in contrast with much of the prevailing literature, see e.g. (Scarf et al., 2022), the scores for both teams are negatively correlated in our model, reflecting teams cancelling each other out. See also related theory in (Malinovsky and Rinott, 2023). Our model is also capable of generating heavy-tailed non-Gaussian behaviour. Here, we present an approximate analysis based on the Variance Gamma distribution (Fischer et al., 2024).

Our empirical application to curling matches is also interesting and important in its own right. We are able to quantify the effect of the LSFE, which serves as an analog to the home-field advantage in other sports (Boudreaux et al., 2017; Ehrlich and Potter, 2023). In curling the LSFE accrues to the team that wins a draw-shot challenge at the start of the match. This is a test of skill, markedly different to e.g. the coin toss at the start of a soccer match. As a result this may reflect teams' overall level of accuracy in idealised settings. Regression results also suggest that much of the discrimination between different teams is due to the quality of their defensive play. Using this insight leads to an interesting way of rating curling teams using bond-rating terminology. It is also interesting that analytical work places such a high value upon good defensive play (McHale et al., 2012).

There remains enduring interest in sports analytics (Baker et al., 2022; Fry et al., 2024; Singh et al., 2023). Sports can also be of interest in pedagogic work (Wooten and White, 2021). Allied to the above, the statistical analysis of historical sporting results can lead to challenging yet informative regression and generalised linear modelling examples, both in terms of the computation and the model interpretation. Some teaching examples on a related theme can be found in (Fry and Burke, 2022).

Appendix A. Proofs

A.1. **Proof to Proposition 1.** From equations (3.2)-(3.3) and using a Gaussian approximation

$$\begin{aligned} Pr(X \text{ wins}) &\approx Pr(X-Y\geq 0.5) \\ &= 1-Pr(X-Y\leq 0.5) \\ &= 1-\Phi\left(\frac{0.5-np_X+np_Y}{\sqrt{np_X(1-p_X)+np_Y(1-p_Y)+2np_Xp_Y}}\right) \\ &= \Phi\left(\frac{np_X-np_Y-0.5}{\sqrt{np_X(1-p_X)+np_Y(1-p_Y)+2np_Xp_Y}}\right) \end{aligned}$$

Similarly, the probability of a draw can be calculated as

$$\begin{aligned} Pr(-0.5\leq X-Y\leq 0.5) &\approx \Phi\left(\frac{0.5-np_X+np_Y}{\sqrt{np_X(1-p_X)+np_Y(1-p_Y)+2np_Xp_Y}}\right) \\ &- \Phi\left(\frac{-0.5-np_X+np_Y}{\sqrt{np_X(1-p_X)+np_Y(1-p_Y)+2np_Xp_Y}}\right) \\ &= \Phi\left(\frac{0.5-np_X+np_Y}{\sqrt{np_X(1-p_X)+np_Y(1-p_Y)+2np_Xp_Y}}\right) \\ &+ \Phi\left(\frac{0.5+np_X-np_Y}{\sqrt{np_X(1-p_X)+np_Y(1-p_Y)+2np_Xp_Y}}\right)-1. \end{aligned}$$

A.2. **Proof to Proposition 2.** Cases 1-2 follow from Proposition 1 with $n = 80$. For case 3, if the two teams are level at the end of 10 ends, the winning stone rolled is allocated to Team X with probability $\frac{p_X}{p_X+p_Y}$.

A.3. **Proof to Proposition 3.** We have that $f(n)=\frac{n^{\lambda-1}\left(\frac{\lambda}{\hat{n}}\right)^{\lambda}e^{-\left(\frac{\lambda}{\hat{n}}\right)n}}{\Gamma(\lambda)}$. Conditional on n we have that

$$f(x|n)=\frac{1}{\sqrt{2\pi}\sigma n^{\frac{1}{2}}}e^{-\frac{(x-n\mu)^2}{2\sigma^2}}.$$

Next, integrating out the unobserved n gives

$$\begin{aligned} f(x) &= \int_0^{\infty}\frac{n^{\lambda-1}\left(\frac{\lambda}{\hat{n}}\right)^{\lambda}e^{-\left(\frac{\lambda}{\hat{n}}\right)n}}{\Gamma(\lambda)}\frac{1}{\sqrt{2\pi}\sigma n^{\frac{1}{2}}}e^{-\frac{(x-n\mu)^2}{2n\sigma^2}}dn \\ &= \frac{\left(\frac{\lambda}{\hat{n}}\right)^{\lambda}\exp\left\{\frac{\mu x}{\sigma^2}\right\}}{\sqrt{2\pi}\sigma\Gamma(\lambda)}\int_0^{\infty}n^{\lambda-\frac{1}{2}-1}\exp\left\{-\frac{x^2}{\sigma^2}\cdot\frac{1}{2n}-\left(\frac{2\lambda}{\hat{n}}+\frac{\mu^2}{\sigma^2}\right)\cdot\frac{n}{2}\right\}dn\,. \end{aligned} \tag{A.1}$$

Next, using the following integration formula (Bingham and Kiesel, 2001):

$$\int_0^{\infty}x^{\lambda-1}\exp\left\{-\frac{1}{2}\left(\gamma^2x+\frac{\delta^2}{x}\right)\right\}dx=2K_{\lambda}(\delta\gamma)\gamma^{-\lambda}\delta^{\lambda}\,,$$

it follows that the integral in (A.1) becomes

$$2K_{\lambda-\frac{1}{2}}\left(\frac{|x|}{\sigma}\sqrt{\frac{2\lambda}{\hat{n}}+\frac{\mu^2}{\sigma^2}}\right)\left(\frac{|x|}{\sigma}\right)^{\lambda-\frac{1}{2}}\left(\frac{2\lambda}{\hat{n}}+\frac{\mu^2}{\sigma^2}\right)^{\frac{1}{4}-\frac{\lambda}{2}}. \tag{A.2}$$

Combining equations (A.1-A.2) it follows that the probability density function can be written as

$$\frac{\left(\frac{\lambda}{\hat{n}}\right)^{\lambda}\sqrt{2}\exp\left\{\frac{\mu x}{\sigma^2}\right\}}{\sqrt{\pi}\sigma\Gamma(\lambda)}K_{\lambda-\frac{1}{2}}\left(\frac{|x|}{\sigma}\sqrt{\frac{2\lambda}{\hat{n}}+\frac{\mu^2}{\sigma^2}}\right)\left(\frac{|x|}{\sigma}\right)^{\lambda-\frac{1}{2}}\left(\frac{2\lambda}{\hat{n}}+\frac{\mu^2}{\sigma^2}\right)^{\frac{1}{4}-\frac{\lambda}{2}}. \tag{A.3}$$

Comparing with the probability density function given in Definition 1 the stated result follows.

A.4. **Proof to Proposition 4.** The distributional result for the score difference is a special case of Proposition 3 with parameters $\mu = p_X - p_Y$ and $\sigma^2 = p_X(1-p_X) + p_Y(1-p_Y) + 2p_Xp_Y$.

References

R. Baker, S. Chadwick, R. Parma, and P. Scarf. The binomial-match, outcome uncertainty, and the case of netball. *J. of the Operat. Res. Soc.*, 73:1856–1872, 2022.

N. H. Bingham and J. M. Fry. *Regression: Linear models in statistics.* Springer, London Dordtrecht Heidelberg New York, 2010.

N. H. Bingham and R. Kiesel. Semi-parametric modelling in finance: theoretical foundations. *Quant. Financ.*, 1:1–10, 2001.

C. J. Boudreaux, S. J. Sanders, and B. Walia. A natural experiment to determine the crowd effect upon home court advantage. *J. of Sports Econ.*, 18:737–749, 2017.

J. Ehrlich and J. Potter. Estimating the effect of attendance on home advantage in the national basketball association. *Appl. Econ. Lett.*, 30:1471–1482, 2023.

R. Finlay and E. Seneta. Stationary-increment student and variance-gamma processes. *J. of Appl. Prob.*, 43:441–453, 2006.

A. Fischer, R. E. Gaunt, and A. Sarantsev. The variance-gamma distribution: A review. *Stat. Sci.*, 2024. forthcoming.

J. Fry, O. Smart, J-P. Serbera, and B. Klar. A variance gamma model for rugby union matches. *J. of Quant. Anal. in Sports*, 17:67–75, 2021.

J. M. Fry and M. Burke. *Quantitative methods in finance using R.* Open University Press, Maidenhead, 2022.

J. M. Fry, T. Brighton, and S. Fanzon. Faster identification of faster formula 1 drivers via time-rank duality. *Econ. Lett.*, 237:111671, 2024.

J. Lawson and M. Rave. Sweeping gestures: A control theory model for curling. *The Coll. Math. J.*, 51:132–140, 2020.

Y. Malinovsky and Y. Rinott. On tournaments and negative dependence. *J. of Appl. Prob.*, 2023. forthcoming.

I. McHale, P. Scarf, and D. E. Folker. On the development of a soccer player performance rating system for the english premier league. *Interfaces*, 42:339–351, 2012.

P. Scarf, R. Parama, and I. McHale. On outcome uncertainty and scoring rates in sport: The case of international rugby union. *Eur. J. of Operat. Res.*, 273:721–730, 2019.

P. Scarf, A. Khare, and N. Alotaibi. On skill and chance in sport. *IMA J. of Manag. Math.*, 33:53–73, 2022.

A. Singh, P. Scarf, and R. Baker. A unified theory for bivariate scores in possessive ball-sports. the case of handball. *Eur. J. of Operat. Res.*, 304:1099–1112, 2023.

K. A. Willoughby and K. J. Kostuk. Preferred scenarios in the sport of curling. *Interfaces*, 34:117–122, 2004.

K. A. Willoughby and K. J. Kostuk. An analysis of a strategic decision in the sport of curling. *Decision Analysis*, 2:58–63, 2005.

J. Wooten and D. R. White. Teaching economics with scenes from moneyball. *J. for Econ. Educ.*, 21: 31–37, 2021.