Numbers Sequences and Series

Revision Guide

Dr. Silvio Fanzon

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Revision Guide

Revision Guide document for the module **Numbers Sequences and Series 400297** 2024/25 at the University of Hull. If you have any question or find any typo, please email me at

S.Fanzon@hull.ac.uk

Full lenght Lecture Notes of the module available at

silviofanzon.com/2024-NSS-Notes

Recommended revision strategy

Make sure you are very comfortable with:

1. The Definitions, Theorems, Proofs, and Examples contained in this Revision Guide
2. The Tutorial and Homework questions
3. The 2023/24 Exam Paper questions.
4. The Checklist below

Checklist

You should be comfortable with the following topics/taks:

Preliminaries

- Prove that $\sqrt{p} \notin \mathbb{Q}$ for p a prime number
-

Complex Numbers

- Sum, multiplication and division of complex numbers
- Computing the complex conjugate
- Computing the inverse of a complex number
- Find modulus and argument of a complex number
- Compute Cartesian, Trigonometric and Exponential form of a complex number
- Complex exponential and its properties
- Computing powers of complex numbers
- Solving degree 2 polynomial equations in $\mathbb{C}$
- Long division of polynomials
- Solving higher degree polynomial equations in $\mathbb{C}$
- Finding the roots of unity
- Finding the n-th roots of a complex number

if they contain the same elements.

1 Preliminaries

Theorem 1.1

The number $\sqrt{2}$ does not belong to $\mathbb{Q}$.

Proof

Aassume by contradiction that

$$\sqrt{2} \in \mathbb{Q}\,. \tag{1.1}$$

1. Therefore, there exists $m, n \in \mathbb{N}$, $n \neq 0$, such that
$$\frac{m}{n} = \sqrt{2}\,.$$
2. **Withouth loss of generality**, we can **assume** that m and n have no common factors.
3. Square the equation to get
$$\frac{m^2}{n^2} = 2 \quad \Longrightarrow \quad m^2 = 2n^2\,. \tag{1.2}$$
Therefore the integer m^2 is an even number.
4. Since m^2 is an even number, it follows that also m is an even number. Then there exists $p \in \mathbb{N}$ such that
$$m = 2p\,. \tag{1.3}$$
5. Substitute (1.3) in (1.2) to get
$$m^2 = 2n^2 \implies (2p)^2 = 2n^2 \implies 4p^2 = 2n^2$$
Dividing both terms by 2, we obtain
$$n^2 = 2p^2\,. \tag{1.4}$$
6. We now make a series of observations:
   - Equation (1.4) says that n^2 is even.
   - The same argument in Step 4 guarantees that also n is even.
   - Therefore n and m are both even, meaning they have 2 as common factor.
   - But Step 2 says that n and m have no common factors. **Contradiction**
7. Our reasoning has run into a **contradiction**, stemming from assumption (1.1). Therefore (1.1) is **FALSE**, and so
$$\sqrt{2} \notin \mathbb{Q}$$
ending the proof.

1.1 Set Theory

Proposition 1.3

Let A and B be sets. Then

$$A = B \quad \Longleftrightarrow \quad A \subseteq B \text{ and } B \subseteq A\,.$$

Definition 1.4

Let Ω be a set, and $A_n \subseteq \Omega$ a family of subsets, where $n \in \mathbb{N}$.

1. The **infinte union** of the A_n is the set
$$\bigcup_{n \in \mathbb{N}} A_n := \{x \in \Omega \, : \, x \in A_n \text{ for at least one } n \in \mathbb{N}\}\,.$$
2. The **infinte intersection** of the A_n is the set
$$\bigcap_{n \in \mathbb{N}} A_n := \{x \in \Omega \, : \, x \in A_n \text{ for all } n \in \mathbb{N}\}\,.$$

Example 1.5

Question. Define $\Omega := \mathbb{N}$ and a family A_n by

$$A_n = \{n, n+1, n+2, n+3, \ldots\}, \quad n \in \mathbb{N}.$$

1. Prove that
$$\bigcup_{n \in \mathbb{N}} A_n = \mathbb{N}. \tag{1.5}$$
2. Prove that
$$\bigcap_{n \in \mathbb{N}} A_n = \emptyset. \tag{1.6}$$

Solution.

1. Assume that $m \in \cup_n A_n$. Then $m \in A_n$ for at least one $n \in \mathbb{N}$. Since $A_n \subseteq \mathbb{N}$, we conclude that $m \in \mathbb{N}$. This shows
$$\bigcup_{n \in \mathbb{N}} A_n \subseteq \mathbb{N}.$$
Conversely, suppose that $m \in \mathbb{N}$. By definition $m \in A_m$. Hence there exists at least one index n, $n = m$ in this case, such that $m \in A_n$. Then by definition $m \in \cup_{n \in \mathbb{N}} A_n$, showing that
$$\mathbb{N} \subseteq \bigcup_{n \in \mathbb{N}} A_n.$$
This proves (1.5).
2. Suppose that (1.6) is false, i.e.,
$$\bigcap_{n \in \mathbb{N}} A_n \neq \emptyset.$$
This means there exists some $m \in \mathbb{N}$ such that $m \in \cap_{n \in \mathbb{N}} A_n$. Hence, by definition, $m \in A_n$ for all $n \in \mathbb{N}$. However $m \notin A_{m+1}$, yielding a contradiction. Thus (1.6) holds.

Definition 1.6

Let $A, B \subseteq \Omega$. The **complement** of A with respect to B is the set of elements of B which do not belong to A, that is

$$B \setminus A := \{x \in \Omega : x \in B \text{ and } x \notin A\}.$$

In particular, the complement of A with respect to Ω is denoted by

$$A^c := \Omega \setminus A := \{x \in \Omega : x \notin A\}.$$

Example 1.7

Question. Suppose $A, B \subseteq \Omega$. Prove that

$$A \subseteq B \iff B^c \subseteq A^c.$$

Solution. Let us prove the above claim:

- First implication $\implies$:
Suppose that $A \subseteq B$. We need to show that $B^c \subseteq A^c$. Hence, assume $x \in B^c$. By definition this means that $x \notin B$. Now notice that we cannot have that $x \in A$. Indeed, assume $x \in A$. By assumption we have $A \subseteq B$, hence $x \in B$. But we had assumed $x \in B$, contradiction. Therefore it must be that $x \notin A$. Thus $B^c \subseteq A^c$.
- Second implication $\impliedby$: Note that, for any set,
$$(A^c)^c = A.$$
Hence, by the first implication,
$$B^c \subseteq A^c \implies (A^c)^c \subseteq (B^c)^c \implies A \subseteq B.$$

Proposition 1.8: De Morgan's Laws

Suppose $A, B \subseteq \Omega$. Then

$$(A \cap B)^c = A^c \cup B^c, \qquad (A \cup B)^c = A^c \cap B^c.$$

Definition 1.9

Let Ω be a set. The **power set** of Ω is

$$\mathscr{P}(\Omega) := \{A : A \subseteq \Omega\}.$$

Example 1.10

Question. Compute the power set of

$$\Omega = \{x, y, z\}.$$

Solution. $\mathscr{P}(\Omega)$ has $2^3 = 8$, and

$$\mathscr{P}(\Omega) = \{\emptyset, \{x\}, \{y\}, \{z\}, \{x, y\} \tag{1.7}$$
$$\{x, z\}, \{y, z\}, \{x, y, z\}\}. \tag{1.8}$$

Definition 1.11

Let A, B be sets. The **product** of A and B is the set of pairs

$$A \times B := \{(a, b) : a \in A, b \in B\}.$$

1.2 Relations

Definition 1.12

Suppose A is a set. A **binary relation** R on A is a subset

$$R \subseteq A \times A .$$

Definition 1.13: Equivalence relation

A binary relation R is called an **equivalence relation** if it satisfies the following properties:

1. **Reflexive**: For each $x \in A$ one has
$$(x, x) \in R ,$$
2. **Symmetric**: We have
$$(x, y) \in R \implies (y, x) \in R$$
3. **Transitive**: We have
$$(x, y) \in R , \; (y, z) \in R \implies (x, z) \in R$$

If $(x, y) \in R$ we write

$$x \sim y$$

and we say that x and y are **equivalent**.

Definition 1.14: Equivalence classes

Suppose R is an **equivalence relation** on A. The **equivalence class** of an element $x \in A$ is the set

$$[x] := \{y \in A : \; y \sim x\} .$$

The set of equivalence classes of elements of A with respect to the equivalence relation R is denoted by

$$A/R := A/\sim := \{[x] : \; x \in A\} .$$

Proposition 1.15

Let $\sim$ be an equivalence relation on A. Then

1. For each $x \in A$ we have
$$[x] \neq \emptyset$$
2. For all $x, y \in A$ it holds
$$x \sim y \iff [x] = [y] .$$

Example 1.16: Equality is an equivalence relation

Question. The equality defines a **binary relation** on $\mathbb{Q} \times \mathbb{Q}$, via

$$R := \{(x, y) \in \mathbb{Q} \times \mathbb{Q} : \; x = y\} .$$

1. Prove that R is an **equivalence relation**.
2. Prove that $[x] = \{x\}$ and compute $\mathbb{Q}/R$.

Solution.

1. We need to check that R satisfies the 3 properties of an equivalence relation:
   - Reflexive: It holds, since $x = x$ for all $x \in \mathbb{Q}$,
   - Symmetric: Again $x = y$ if and only if $y = x$,
   - Transitive: If $x = y$ and $y = z$ then $x = z$.

   Therefore, R is an equivalence relation.
2. The class of equivalence of $x \in \mathbb{Q}$ is given by
$$[x] = \{x\} ,$$
   that is, this relation is quite trivial, given that each element of $\mathbb{Q}$ can only be related to itself. The quotient space is then
$$\mathbb{Q}/R = \{[x] : \; x \in \mathbb{Q}\} = \{\{x\} : \; x \in \mathbb{Q}\} .$$

Example 1.17

Question. Let R be the binary relation on the set $\mathbb{Q}$ of rational numbers defined by

$$x \sim y \iff x - y \in \mathbb{Z} .$$

1. Prove that R is an equivalence relation on $\mathbb{Q}$.
2. Compute $[x]$ for each $x \in \mathbb{Q}$.
3. Compute $\mathbb{Q}/R$.

Solution.

1. We have:
   - Reflexive: Let $x \in \mathbb{Q}$. Then $x - x = 0$ and $0 \in \mathbb{Z}$. Thus $x \sim x$.
   - Symmetric: If $x \sim y$ then $x - y \in \mathbb{Z}$. But then also
$$-(x - y) = y - x \in \mathbb{Z}$$
     and so $y \sim x$.
   - Transitive: Suppose $x \sim y$ and $y \sim z$. Then
$$x - y \in \mathbb{Z} \text{ and } y - z \in \mathbb{Z} .$$

Thus, we have

$$x - z = (x - y) + (y - z) \in \mathbb{Z}$$

showing that $x \sim z$.

Thus, we have shown that R is an equivalence relation on $\mathbb{Q}$.

2. Note that

$$x \sim y \quad \iff \quad \exists\, n \in \mathbb{Z} \text{ s.t. } y = x + n\,.$$

Therefore the equivalence classes with respect to $\sim$ are

$$[x] = \{x + n \,:\, n \in \mathbb{Z}\}\,.$$

Each equivalence class has exactly one element in $[0, 1) \cap \mathbb{Q}$, meaning that:

$$\forall x \in \mathbb{Q}\,, \ \exists!\, q \in \mathbb{Q} \text{ s.t. } 0 \leq q < 1 \text{ and } q \in [x]\,. \tag{1.9}$$

Indeed: take $x \in \mathbb{Q}$ arbitrary. Then $x \in [n, n + 1)$ for some $n \in \mathbb{Z}$. Setting $q := x - n$ we obtain that

$$x = q + n\,, \qquad q \in [0, 1)\,,$$

proving (1.9). In particular (1.9) implies that for each $x \in \mathbb{Q}$ there exists $q \in [0, 1) \cap \mathbb{Q}$ such that

$$[x] = [q]\,.$$

3. From Point 2 we conclude that

$$\mathbb{Q}/R = \{[x] \,:\, x \in \mathbb{Q}\} = \{q \in \mathbb{Q} \,:\, 0 \leq q < 1\}\,.$$

Definition 1.18: Partial order

A binary relation R on A is called a **partial order** if it satisfies the following properties:

1. **Reflexive**: For each $x \in A$ one has

$$(x, x) \in R\,,$$

2. **Antisymmetric**: We have

$$(x, y) \in R \text{ and } (y, x) \in R \implies x = y$$

3. **Transitive**: We have

$$(x, y) \in R\,,\ (y, z) \in R \implies (x, z) \in R$$

Definition 1.19: Total order

A binary relation R on A is called a **total order relation** if it satisfies the following properties:

1. **Partial order**: R is a partial order on A.
2. **Total**: For each $x, y \in A$ we have

$$(x, y) \in R \ \text{ or } \ (y, x) \in R\,.$$

Example 1.20: Set inclusion is a partial order but not total order

Question. Let Ω be a non-empty set and consider its **power set**

$$\mathscr{P}(\Omega) = \{A \,:\, A \subseteq \Omega\}\,.$$

The inclusion defines **binary relation** on $\mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$, via

$$R := \{(A, B) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \,:\, A \subseteq B\}\,.$$

1. Prove that R is an **order relation**.
2. Prove that R is **not a total order**.

Solution.

1. Check that R is a partial order relation on $\mathscr{P}(\Omega)$:
   - Reflexive: It holds, since $A \subseteq A$ for all $A \in \mathscr{P}(\Omega)$.
   - Antisymmetric: If $A \subseteq B$ and $B \subseteq A$, then $A = B$.
   - Transitive: If $A \subseteq B$ and $B \subseteq C$, then, by definition of inclusion, $A \subseteq C$.
2. In general, R is **not** a total order. For example consider

$$\Omega = \{x, y\}\,.$$

Thus

$$\mathscr{P}(\Omega) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}\,.$$

If we pick $A = \{x\}$ and $B = \{y\}$ then $A \cap B = \emptyset$, meaning that

$$A \not\subseteq B\,, \quad B \not\subseteq A\,.$$

This shows R is not a total order.

Example 1.21: Inequality is a total order

Question. Consider the binary relation

$$R := \{(x, y) \in \mathbb{Q} \times \mathbb{Q} \,:\, x \leq y\}\,.$$

Prove that R is a **total order relation**.
Solution. We need to check that:

1. Reflexive: It holds, since $x \leq x$ for all $x \in \mathbb{Q}$,
2. Antisymmetric: If $x \leq y$ and $y \leq x$ then $x = y$.
3. Transitive: If $x \leq y$ and $y \leq z$ then $x \leq z$.

Finally, we halso have that R is a **total order** on $\mathbb{Q}$, since for all $x, y \in \mathbb{Q}$ we have

$$x \leq y \quad \text{or} \quad y \leq x \,.$$

1.3 Induction

Definition 1.22: Principle of Inducion

Let $\alpha(n)$ be a statement which depends on $n \in \mathbb{N}$. Suppose that

1. $\alpha(1)$ is true, and
2. Whenever $\alpha(n)$ is true, then $\alpha(n+1)$ is true.

Then $\alpha(n)$ is true for all $n \in \mathbb{N}$.

Example 1.23: Formula for summing first n natural numbers

Question. Prove by induction that the following formula holds for all $n \in \mathbb{N}$:

$$1 + 2 + 3 + \ldots + (n-1) + n = \frac{n(n+1)}{2} \,. \tag{1.10}$$

Solution. Define

$$S(n) = 1 + 2 + \ldots + n \,.$$

This way the formula at (1.10) is equivalent to

$$S(n) = \frac{n(n+1)}{2} \,, \quad \forall\, n \in \mathbb{N} \,.$$

1. It is immediate to check that (1.10) holds for $n = 1$.
2. Suppose (1.10) holds for $n = k$. Then

$$\begin{aligned} S(k+1) &= 1 + \ldots + k + (k+1) && (1.11)\\ &= S(k) + (k+1) && (1.12)\\ &= \frac{k(k+1)}{2} + (k+1) && (1.13)\\ &= \frac{k(k+1) + 2(k+1)}{2} && (1.14)\\ &= \frac{(k+1)(k+2)}{2} && (1.15) \end{aligned}$$

where in the first equality we used that (1.10) holds for $n = k$. We have proven that

$$S(k+1) = \frac{(k+1)(k+2)}{2} \,.$$

The RHS in the above expression is exactly the RHS of (1.10) computed at $n = k + 1$. Therefore, we have shown that formula (1.10) holds for $n = k + 1$.

By the Principle of Induction, we conclude that (1.10) holds for all $n \in \mathbb{N}$.

Example 1.24: Bernoulli's inequality

Question. Let $x \in \mathbb{R}$ with $x > -1$. Bernoulli's inequality states that

$$(1+x)^n \geq 1 + nx \,, \quad \forall\, n \in \mathbb{N} \,. \tag{1.16}$$

Prove Bernoulli's inequality by induction.
Solution. Let $x \in \mathbb{R}, x > -1$. We prove the statement by induction:

- Base case: (1.16) holds with equality when $n = 1$.
- Induction hypothesis: Let $k \in \mathbb{N}$ and suppose that (1.16) holds for $n = k$, i.e.,

$$(1+x)^k \geq 1 + kx \,.$$

Then

$$\begin{aligned} (1+x)^{k+1} &= (1+x)^k(1+x)\\ &\geq (1+kx)(1+x)\\ &= 1 + kx + x + kx^2\\ &\geq 1 + (k+1)x \,, \end{aligned}$$

where we used that $kx^2 \geq 0$. Then (1.16) holds for $n = k + 1$.

By induction we conclude (1.16).

1.4 Absolute value

Definition 1.25: Absolute value

For $x \in \mathbb{R}$ we define its **absolute value** as the quantity

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Proposition 1.26

For all $x \in \mathbb{R}$ they hold:

1. $|x| \geq 0$.
2. $|x| = 0$ if and only if $x = 0$.
3. $|x| = |-x|$.

Lemma 1.27

Let $x, y \in \mathbb{R}$. Then

$$|x| \leq y \iff -y \leq x \leq y\,.$$

Corollary 1.28

Let $x, y \in \mathbb{R}$. Then

$$|x| < y \iff -y < x < y\,.$$

Theorem 1.29: Triangle inequality

For every $x, y \in \mathbb{R}$ we have

$$||x| - |y|| \leq |x + y| \leq |x| + |y|\,. \tag{1.17}$$

Proposition 1.30

For any $x, y \in \mathbb{R}$ it holds

$$||x| - |y|| \leq |x - y| \leq |x| + |y|\,. \tag{1.18}$$

Moreover for any $x, y, z \in \mathbb{R}$ it holds

$$|x - y| \leq |x - z| + |z - y|\,.$$

2 Real Numbers

2.1 Fields

Definition 2.1: Binary operation

A binary operation on a set K is a function

$$\circ \;:\; K \times K \to K$$

which maps the ordered pair (x, y) into $x \circ y$.

Definition 2.2

Let K be a set and $\circ \;:\; K \times K \to K$ be a binary operation on K. We say that:

1. $\circ$ is **commutative** if

$$x \circ y = y \circ x\,, \quad \forall\, x, y \in K$$

2. $\circ$ is **associative** if

$$(x \circ y) \circ z = x \circ (y \circ z)\,, \quad \forall\, x, y, z \in K$$

3. An element $e \in K$ is called **neutral element** of $\circ$ if

$$x \circ e = e \circ x = x\,, \quad \forall\, x \in K$$

4. Let e be a neutral element of $\circ$ and let $x \in K$. An element $y \in K$ is called an **inverse** of x with respect to $\circ$ if

$$x \circ y = y \circ x = e\,.$$

Example 2.3

Question. Let $K = \{0, 1\}$ be a set with binary operation $\circ$ defined by the table

$\circ$	0	1
0	1	1
1	0	0

1. Is $\circ$ commutative? Justify your answer.
2. Is $\circ$ associative? Justify your answer.

Solution.

1. We have

$$0 \circ 1 = 1\,, \quad 1 \circ 0 = 0$$

and therefore

$$0 \circ 1 \neq 1 \circ 0\,.$$

showing that $\circ$ is not commutative.

2. We have

$$(0 \circ 1) \circ 1 = 1 \circ 1 = 0\,,$$

while

$$0 \circ (1 \circ 1) = 0 \circ 0 = 1\,,$$

so that

$$(0 \circ 1) \circ 1 \neq 0 \circ (1 \circ 1)\,.$$

Thus, $\circ$ is not associative.

Definition 2.4: Field

Let K be a set with binary operations of **addition**

$$+ \;:\; K \times K \to K\,, \quad (x, y) \mapsto x + y$$

and **multiplication**

$$\cdot \;:\; K \times K \to K\,, \quad (x, y) \mapsto x \cdot y = xy\,.$$

We call the triple $(K, +, \cdot)$ a **field** if:

1. The addition $+$ satisfies: $\forall\, x, y, z \in K$
   - (A1) **Commutativity and Associativity**:
   $$x + y = y + x$$
   $$(x + y) + z = x + (y + z)$$
   - (A2) **Additive Identity**: There exists a **neutral element** in K for $+$, which we call 0. It holds:
   $$x + 0 = 0 + x = x$$
   - (A3) **Additive Inverse**: There exists an **inverse** of x with respect to $+$. We call this element the **additive inverse** of x and denote it by $-x$. It holds
   $$x + (-x) = (-x) + x = 0$$
2. The multiplication $\cdot$ satisifes: $\forall\, x, y, z \in K$
   - (M1) **Commutativity and Associativity**:
   $$x \cdot y = y \cdot x$$
   $$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

- (M2) **Multiplicative Identity**: There exists a **neutral element** in K for $\cdot$, which we call 1. It holds:
$$x \cdot 1 = 1 \cdot x = x$$
- (M3) **Multiplicative Inverse**: If $x \neq 0$ there exists an **inverse** of x with respect to $\cdot$. We call this element the **multiplicative inverse** of x and denote it by x^{-1}. It holds
$$x \cdot x^{-1} = x^{-1} \cdot x = 1$$

3. The operations $+$ and $\cdot$ are related by
   - (AM) **Distributive Property**: $\forall\, x, y, z \in K$
$$x \cdot (y + z) = (x \cdot y) + (y \cdot z)\,.$$

Theorem 2.5

Consider the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ with the usual operations $+$ and $\cdot$. We have:

- $(\mathbb{N}, +, \cdot)$ is **not a field**.
- $(\mathbb{Z}, +, \cdot)$ is **not a field**.
- $(\mathbb{Q}, +, \cdot)$ **is a field**.

Theorem 2.6

Let K with $+$ and $\cdot$ defined by

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

Then $(K, +, \cdot)$ is a field.

Proposition 2.7: Uniqueness of neutral elements and inverses

Let $(K, +, \cdot)$ be a field. Then

1. There is a unique element in K with the property of 0.
2. There is a unique element in K with the property of 1.
3. For all $x \in K$ there is a unique additive inverse $-x$.
4. For all $x \in K$, $x \neq 0$, there is a unique multiplicative inverse x^{-1}.

Proof

1. Suppose that $0 \in K$ and $\tilde{0} \in K$ are both neutral element of $+$, that is, they both satisfy (A2). Then
$$0 + \tilde{0} = 0$$
since $\tilde{0}$ is a neutral element for $+$. Moreover
$$\tilde{0} + 0 = \tilde{0}$$
since 0 is a neutral element for $+$. By commutativity of $+$, see property (A1), we have
$$0 = 0 + \tilde{0} = \tilde{0} + 0 = \tilde{0}\,,$$
showing that $0 = \tilde{0}$. Hence the neutral element for $+$ is unique.
2. Exercise.
3. Let $x \in K$ and suppose that $y, \tilde{y} \in K$ are both additive inverses of x, that is, they both satisfy (A3). Therefore
$$x + y = 0$$
since y is an additive inverse of x and
$$x + \tilde{y} = 0$$
since $\tilde{y}$ is an additive inverse of x. Therefore we can use commutativity and associativity and of $+$, see property (A1), and the fact that 0 is the neutral element of $+$, to infer
$$\begin{aligned} y &= y + 0 = y + (x + \tilde{y}) \\ &= (y + x) + \tilde{y} = (x + y) + \tilde{y} \\ &= 0 + \tilde{y} = \tilde{y}\,, \end{aligned}$$
concluding that $y = \tilde{y}$. Thus there is a unique additive inverse of x, and
$$y = \tilde{y} = -x\,,$$
with $-x$ the element from property (A3).
4. Exercise.

Definition 2.8

Let K be a set with binary operations $+$ and $\cdot$, and with an order relation $\leq$. We call $(K, +, \cdot, \leq)$ an **ordered field** if:

1. $(K, +, \cdot)$ is a field
2. There $\leq$ is of **total order** on K: $\forall\, x, y, z \in K$
   - (O1) **Reflexivity**:
$$x \leq x$$

- (O2) **Antisymmetry**:

$$x \leq y \text{ and } y \leq x \implies x = y$$

- (O3) **Transitivity**:

$$x \leq y \text{ and } y \leq z \implies x = z$$

- (O4) **Total order**:

$$x \leq y \text{ or } y \leq x$$

3. The operations $+$ and $\cdot$, and the total order $\leq$, are related by the following properties: $\forall x, y, z \in K$

- (AM) **Distributive**: Relates addition and multiplication via

$$x \cdot (y + z) = x \cdot y + x \cdot z$$

- (AO) Relates addition and order with the requirement:

$$x \leq y \implies x + z \leq y + z$$

- (MO) Relates multiplication and order with the requirement:

$$x \geq 0, \, y \geq 0 \implies x \cdot y \geq 0$$

Theorem 2.9

$(\mathbb{Q}, +, \cdot, \leq)$ is an **ordered field**.

2.2 Supremum and infimum

Definition 2.10: Upper bound, bounded above, supremum, maximum

Let $A \subseteq K$:

1. We say that $b \in K$ is an **upper bound** for A if

$$a \leq b \,, \quad \forall a \in A \,.$$

2. We say that A is **bounded above** if there exists and upper bound $b \in K$ for A.

3. We say that $s \in K$ is the **least upper bound** or **supremum** of A if:

- s is an upper bound for A,
- s is the smallest upper bound of A, that is,

$$\text{If } b \in K \text{ is upper bound for } A \text{ then } s \leq b \,.$$

If it exists, the supremum is denoted by

$$s = \sup A \,.$$

4. Let $A \subseteq K$. We say that $M \in K$ is the **maximum** of A if:

$$M \in A \text{ and } a \leq M \,, \, \forall a \in A \,.$$

If it exists, we denote the maximum by

$$M = \max A \,.$$

Remark 2.11

Note that if a set $A \subseteq K$ in **NOT** bounded above, then the supremum does not exist, as there are no upper bounds of A.

Proposition 2.12: Relationship between Max and Sup

Let $A \subseteq K$. If the maximum of A exists, then also the supremum exists, and

$$\sup A = \max A \,.$$

Definition 2.13: Upper bound, bounded below, infimum, minimum

Let $A \subseteq K$:

1. We say that $l \in K$ is a **lower bound** for A if

$$l \leq a \,, \quad \forall a \in A \,.$$

2. We say that A is **bounded below** if there exists a lower bound $l \in K$ for A.

3. We say that $i \in K$ is the **greatest lower bound** or **infimum** of A if:

- i is a lower bound for A,
- i is the largest lower bound of A, that is,

$$\text{If } l \in K \text{ is a lower bound for } A \text{ then } l \leq i \,.$$

If it exists, the infimum is denoted by

$$i = \inf A \,.$$

4. We say that $m \in K$ is the **minimum** of A if:

$$m \in A \text{ and } m \leq a \,, \, \forall a \in A \,.$$

If it exists, we denote the minimum by

$$m = \min A\,.$$

Proposition 2.14

Let $A \subseteq K$. If the minimum of A exists, then also the infimum exists, and

$$\inf A = \min A\,.$$

Proposition 2.15

Let $A \subseteq K$. If $\inf A$ and $\sup A$ exist, then

$$\inf A \le a \le \sup A\,, \quad \forall\, a \in A\,.$$

Proposition 2.16: Relationship between sup and inf

Let $A \subseteq K$. Define

$$-A := \{-a \,:\, a \in A\}\,.$$

They hold

1. If $\sup A$ exists, then $\inf A$ exists and
$$\inf(-A) = -\sup A\,.$$
2. If $\inf A$ exists, then $\sup A$ exists and
$$\sup(-A) = -\inf A\,.$$

2.3 Axioms of Real Numbers

Definition 2.17: Completeness

Let $(K, +, \cdot, \le)$ be an ordered field. We say that K is **complete** if it holds the property:

- (AC) For every $A \subseteq K$ non-empty and bounded above
$$\sup A \in K\,.$$

Theorem 2.18

$\mathbb{Q}$ is not complete. In particular, there exists a set $A \subseteq \mathbb{Q}$ such that

- A is non-empty,
- A is bounded above,
- $\sup A$ does not exist in $\mathbb{Q}$.

Proposition 2.19

Let $(K, +, \cdot, \le)$ be a complete ordered field. Suppose that

Definition 2.20: System of Real Numbers $\mathbb{R}$

A system of Real Numbers is a set $\mathbb{R}$ with two operations $+$ and $\cdot$, and a total order relation $\le$, such that

- $(\mathbb{R}, +, \cdot, \le)$ is an ordered field
- $\mathbb{R}$ sastisfies the Axiom of Completeness

2.3.1 Inductive sets

Definition 2.21: Inductive set

Let $S \subseteq \mathbb{R}$. We say that S is an inductive set if they are satisfied:

Example 2.22

Question. Prove the following:

1. $\mathbb{R}$ is an inductive set.
2. The set $A = \{0, 1\}$ is not an inductive set.

Solution.

1. We have that $1 \in \mathbb{R}$ by axiom (M2). Moreover $(x + 1) \in \mathbb{R}$ for every $x \in \mathbb{R}$, by definition of sum $+$.
2. We have $1 \in A$, but $(1 + 1) \notin A$, since $1 + 1 \neq 0$

Proposition 2.23

Let $\mathscr{M}$ be a collection of inductive subsets of $\mathbb{R}$. Then

$$S := \bigcap_{M \in \mathscr{M}} M$$

is an inductive subset of $\mathbb{R}$.

Definition 2.24: Set of Natural Numbers

Let $\mathscr{M}$ be the collection of **all** inductive subsets of $\mathbb{R}$. We define the set of natural numbers in $\mathbb{R}$ as

$$\mathbb{N} := \bigcap_{M \in \mathscr{M}} M .$$

Proposition 2.25: $\mathbb{N}_{\mathbb{R}}$ is the smallest inductive subset of $\mathbb{R}$

Let $C \subseteq \mathbb{R}$ be an inductive subset. Then

$$\mathbb{N} \subseteq C .$$

In other words, $\mathbb{N}$ is the smallest inductive set in $\mathbb{R}$.

Theorem 2.26

Let $x \in \mathbb{N}$. Then

$$x \geq 1 .$$

3 Properties of $\mathbb{R}$

Theorem 3.1: Archimedean Property

Let $x \in \mathbb{R}$ be given. Then:

1. There exists $n \in \mathbb{N}$ such that
$$n > x \,.$$
2. Suppose in addition that $x > 0$. There exists $n \in \mathbb{N}$ such that
$$\frac{1}{n} < x \,.$$

Theorem 3.2: Archimedean Property (Alternative formulation)

Let $x, y \in \mathbb{R}$, with $0 < x < y$. There exists $n \in \mathbb{N}$ such that
$$nx > y \,.$$

Theorem 3.3: Nested Interval Property

For each $n \in \mathbb{N}$ assume given a closed interval
$$I_n := [a_n, b_n] = \{x \in \mathbb{R} : \; a_n \leq x \leq b_n\} \,.$$
Suppose that the intervals are nested, that is,
$$I_n \supset I_{n+1} \,, \quad \forall \, n \in \mathbb{N} \,.$$
Then
$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset \,. \tag{3.1}$$

Example 3.4

Question. Consider the **open** intervals
$$I_n := \left(0, \frac{1}{n}\right) \,.$$
These are clearly nested
$$I_n \supset I_{n+1} \,, \quad \forall \, n \in \mathbb{N} \,.$$
Prove that
$$\bigcap_{n=1}^{\infty} I_n = \emptyset \,. \tag{3.2}$$

Solution. Suppose by contradiction that the intersection is non-empty. Then there exists $x \in \mathbb{N}$ such that
$$x \in I_n \,, \quad \forall \, n \in \mathbb{N} \,.$$
By definition of I_n the above reads
$$0 < x < \frac{1}{n} \,, \quad \forall \, n \in \mathbb{N} \,. \tag{3.3}$$
Since $x > 0$, by the Archimedean Property in Theorem 3.1 Point 2, there exists $n_0 \in \mathbb{N}$ such that
$$0 < \frac{1}{n_0} < x \,.$$
The above contradicts (3.3). Therefore (3.2) holds.

3.1 Revisiting Sup and Inf

Proposition 3.5: Characterization of Supremum

Let $A \subseteq \mathbb{R}$ be a non-empty set. Suppose that $s \in \mathbb{R}$ is an upper bound for A. They are equivalent:

1. $s = \sup A$
2. For every $\varepsilon > 0$ there exists $x \in A$ such that
$$s - \varepsilon < x \,.$$

Proposition 3.6: Characterization of Infimum

Let $A \subseteq \mathbb{R}$ be a non-empty set. Suppose that $i \in \mathbb{R}$ is a lower bound for A. They are equivalent:

1. $i = \inf A$
2. For every $\varepsilon \in \mathbb{R}$, with $\varepsilon > 0$, there exists $x \in A$ such that
$$x < i + \varepsilon \,.$$

Proposition 3.7

Let $a, b \in \mathbb{R}$ with $a < b$. Let
$$A := (a, b) = \{x \in \mathbb{R} : \; a < x < b\} \,.$$

Then
$$\inf A = a\,, \quad \sup A = b\,.$$

Corollary 3.8

Let $a, b \in \mathbb{R}$ with $a < b$. Let
$$A := (a,b) = \{x \in \mathbb{R} \,:\, a < x < b\}\,.$$
Then $\min A$ and $\max A$ do not exist.

Corollary 3.9

Let $a, b \in \mathbb{R}$ with $a < b$. Let
$$A := [a,b) = \{x \in \mathbb{R} \,:\, a \leq x < b\}\,.$$
Then
$$\min A = \inf A = a\,, \quad \sup A = b\,,$$
$\max A$ does not exist.

Proposition 3.10

Define the set
$$A := \left\{\frac{1}{n} \,:\, n \in \mathbb{N}\right\}.$$
Then
$$\inf A = 0\,, \quad \sup A = \max A = 1\,.$$

Proof

Part 1. We have
$$\frac{1}{n} \leq 1\,, \quad \forall\, n \in \mathbb{N}\,.$$
Therefore 1 is an upper bound for A. Since $1 \in A$, by definition of maximum we conclude that
$$\max A = 1\,.$$
Since the maximum exists, we conclude that also the supremum exists, and
$$\sup A = \max A = 1\,.$$
Part 2. We have
$$\frac{1}{n} > 0\,, \quad \forall\, n \in \mathbb{N}\,,$$
showing that 0 is a lower bound for A. Suppose by contradiction that 0 is not the infimum. Therefore 0 is not the largest lower bound. Then there exists $\varepsilon \in \mathbb{R}$ such that:

- ε is a lower bound for A, that is,
$$\varepsilon \leq \frac{1}{n}\,, \quad \forall\, n \in \mathbb{N}\,, \tag{3.4}$$
- ε is larger than 0:
$$0 < \varepsilon\,.$$

As $\varepsilon > 0$, by the Archimedean Property there exists $n_0 \in \mathbb{N}$ such that
$$0 < \frac{1}{n_0} < \varepsilon\,.$$
This contradicts (3.4). Thus 0 is the largest lower bound of A, that is, $0 = \inf A$.
Part 3. We have that $\min A$ does not exist. Indeed suppose by contradiction that $\min A$ exists. Then
$$\min A = \inf A\,.$$
As $\inf A = 0$ by Part 2, we conclude $\min A = 0$. As $\min A \in A$, we obtain $0 \in A$, which is a contradiction.

3.2 Cardinality

Definition 3.11: Cardinality, Finite, Countable, Uncountable

Let X be a set. The **cardinality** of X is the number of elements in X. We denote the cardinality of X by
$$|X| := \#\ \text{of elements in}\ X\,.$$
Further, we say that:

1. X is **finite** if there exists a natural number $n \in \mathbb{N}$ and a bijection
$$f : \{1, 2, \dots, n\} \to X\,.$$
In particular
$$|X| = n\,.$$
2. X is **countable** if there exists a bijection
$$f : \mathbb{N} \to X\,.$$
In this case we denote the cardinality of X by
$$|X| = |\mathbb{N}|\,.$$
3. X is **uncountable** if X is neither finite, nor countable.

Proposition 3.12

Let X be a countable set and $A \subseteq X$. Then either A is finite or countable.

Example 3.13

Question. Prove that $X = \{a, b, c\}$ is finite.
Solution. Set $Y = \{1, 2, 3\}$. The function $f : X \to Y$ defined by

$$f(1) = a\,, \quad f(2) = b\,, \quad f(3) = c\,,$$

is bijective. Therefore X is finite, with $|X| = 3$.

Example 3.14

Question. Prove that the set of natural numbers $\mathbb{N}$ is countable.
Solution. The function $f : X \to \mathbb{N}$ defined by

$$f(n) := n\,,$$

is bijective. Therefore $X = \mathbb{N}$ is countable.

Example 3.15

Question. Let X be the set of even numbers

$$X = \{2n : n \in \mathbb{N}\}\,.$$

Prove that X is countable.
Solution. Define the map $f : \mathbb{N} \to X$ by

$$f(n) := 2n\,.$$

We have that:

1. f is injective, because

$$f(m) = f(k) \implies 2m = 2k \quad m = k\,.$$

2. f is surjective: Suppose that $m \in X$. By definition of X, there exists $n \in \mathbb{N}$ such that $m = 2n$. Therefore, $f(n) = m$.

We have shown that f is bijective. Thus, X is countable.

Example 3.16

Question. Prove that the set of integers $\mathbb{Z}$ is countable.

Solution. Define $f : \mathbb{N} \to \mathbb{Z}$ by

$$f(n) := \begin{cases} \dfrac{n}{2} & \text{if } n \text{ even} \\ -\dfrac{n+1}{2} & \text{if } n \text{ odd} \end{cases}$$

For example

$$\begin{aligned} &f(0) = 0\,, \quad f(1) = -1\,, \quad f(2) = 1\,, \quad f(3) = -2\,, \\ &f(4) = 2\,, \quad f(5) = -3\,, \quad f(6) = 3\,, \quad f(7) = -4\,. \end{aligned}$$

We have:

1. f is injective: Indeed, suppose that $m \neq n$. If n and m are both even or both odd we have, respectively

$$f(m) = \frac{m}{2} \neq \frac{n}{2} = f(n)$$
$$f(m) = -\frac{m+1}{2} \neq -\frac{n+1}{2} = f(n)\,.$$

If instead m is even and n is odd, we get

$$f(m) = \frac{m}{2} \neq -\frac{n+1}{2} = f(n)\,,$$

since the LHS is positive and the RHS is negative. The case when m is odd and n even is similar.

2. f is surjective: Let $z \in \mathbb{Z}$. If $z \geq 0$, then $m := 2z$ belongs to $\mathbb{N}$, is even, and

$$f(m) = f(2z) = z\,.$$

If instead $z < 0$, then $m := -2z - 1$ belongs to $\mathbb{N}$, is odd, and

$$f(m) = f(-2z-1) = z\,.$$

Therefore f is bijective, showing that $\mathbb{Z}$ is countable.

Proposition 3.17

Let the set A_n be countable for all $n \in \mathbb{N}$. Define

$$A = \bigcup_{n \in \mathbb{N}} A_n\,.$$

Then A is countable.

Theorem 3.18: $\mathbb{Q}$ is countable

The set of rational numbers $\mathbb{Q}$ is countable.

Theorem 3.19: $\mathbb{R}$ is uncountable

The set of Real Numbers $\mathbb{R}$ is **uncountable**.

Theorem 3.20

The set of irrational numbers

$$\mathcal{I} := \mathbb{R} \setminus \mathbb{Q}$$

is uncountable.

Proof

We know that $\mathbb{R}$ in uncountable and $\mathbb{Q}$ is countable. Suppose by contradiction that $\mathcal{I}$ is countable. Then

$$\mathbb{Q} \cup \mathcal{I}$$

is countable by Proposition 3.17, being union of countable sets. Since by definition

$$\mathbb{R} = \mathbb{Q} \cup \mathcal{I} ,$$

we conclude that $\mathbb{R}$ is countable. Contradiction.

4 Complex Numbers

Definition 4.1: Complex Numbers

The set of complex numbers $\mathbb{C}$ is defined as

$$\mathbb{C} := \mathbb{R} + i\mathbb{R} := \{x + iy \,:\, x, y \in \mathbb{R}\}.$$

For a complex number

$$z = x + iy \in \mathbb{C}$$

we say that

- x is the **real part** of z, and denote it by
$$x = \mathrm{Re}(z)$$
- y is the **imaginary part** of z, and denote it by
$$y = \mathrm{Im}(z)$$

We say that

- If $\mathrm{Re}\, z = 0$ then z is a **purely imaginary** number.
- If $\mathrm{Im}\, z = 0$ then z is a **real** number.

Definition 4.2: Addition and multiplication in $\mathbb{C}$

Let $z_1, z_2 \in \mathbb{C}$, so that

$$z_1 = x_1 + iy_1 \,, \quad z_2 = x_2 + iy_2 \,,$$

for some $x_1, x_2, y_1, y_2 \in \mathbb{R}$:

1. The sum of z_1 and z_2 is
$$z_1 + z_2 := (x_1 + x_2) + i\,(y_1 + y_2) \,.$$
2. The multiplication of z_1 and z_2 is
$$z_1 \cdot z_2 := (x_1 \cdot x_2 - y_1 \cdot y_2) + i\,(x_1 \cdot y_2 + x_2 \cdot y_1) \,,$$

Example 4.3

Question. Compute zw, where

$$z = -2 + 3i \,, \quad w = 1 - i \,.$$

Solution. Using the definition we compute

$$\begin{aligned} z \cdot w &= (-2 + 3i) \cdot (1 - i) \\ &= (-2 - (-3)) + (2 + 3)i \\ &= 1 + 5i \,. \end{aligned}$$

Alternatively, we can proceed formally: We just need to recall that i^2 has to be replaced with -1:

$$\begin{aligned} z \cdot w &= (-2 + 3i) \cdot (1 - i) \\ &= -2 + 2i + 3i - 3i^2 \\ &= (-2 + 3) + (2 + 3)i \\ &= 1 + 5i \,. \end{aligned}$$

Proposition 4.4: Additive inverse in $\mathbb{C}$

The neutral element of addition in $\mathbb{C}$ is the number

$$0 := 0 + 0i \,.$$

For any $z = x + iy \in \mathbb{C}$, the unique additive inverse is given by

$$-z := -x - iy \,.$$

Proposition 4.5: Multiplicative inverse in $\mathbb{C}$

The neutral element of multiplication in $\mathbb{C}$ is the number

$$1 := 1 + 0i \,.$$

For any $z = x + iy \in \mathbb{C}$, the unique multiplicative inverse is given by

$$z^{-1} := \frac{x}{x^2 + y^2} + i\,\frac{-y}{x^2 + y^2} \,.$$

Proof

It is immediate to check that 1 is the neutral element of multiplication in $\mathbb{C}$. For the remaining part of the statement, set

$$w := \frac{x}{x^2 + y^2} + i\,\frac{-y}{x^2 + y^2} \,.$$

We need to check that $z \cdot w = 1$

$$\begin{aligned} z \cdot w &= (x+iy) \cdot \left(\frac{x}{x^2+y^2} + i \frac{-y}{x^2+y^2} \right) \\ &= \left(\frac{x^2}{x^2+y^2} - \frac{y \cdot (-y)}{x^2+y^2} \right) + i \left(\frac{x \cdot (-y)}{x^2+y^2} + \frac{xy}{x^2+y^2} \right) \\ &= 1\,, \end{aligned}$$

so indeed $z^{-1} = w$.

Example 4.6

Question. Let $z = 3 + 2i$. Compute z^{-1}.
Solution. By the formula in Propostion 4.5 we immediately get

$$z^{-1} = \frac{3}{3^2+2^2} + \frac{-2}{3^2+2^2} i = \frac{3}{13} - \frac{2}{13} i\,.$$

Alternatively, we can proceed formally:

$$\begin{aligned} (3+2i)^{-1} &= \frac{1}{3+2i} \\ &= \frac{1}{3+2i} \frac{3-2i}{3-2i} \\ &= \frac{3-2i}{3^2+2^2} \\ &= \frac{3}{13} - \frac{2}{13} i\,, \end{aligned}$$

and obtain the same result.

Theorem 4.7

$(\mathbb{C}, +, \cdot)$ is a field.

Example 4.8

Question. Let $w = 1 + i$ and $z = 3 - i$. Compute $\frac{w}{z}$.
Solution. We compute w/z using the two options we have:

1. Using the formula for the inverse from Proposition 4.5 we compute

$$\begin{aligned} z^{-1} &= \frac{x}{x^2+y^2} + i \frac{-y}{x^2+y^2} \\ &= \frac{3}{3^2+1^2} - i \frac{-1}{3^2+1^2} \\ &= \frac{3}{10} + \frac{1}{10} i \end{aligned}$$

and therefore

$$\begin{aligned} \frac{w}{z} &= w \cdot z^{-1} \\ &= (1+i) \left(\frac{3}{10} + \frac{1}{10} i \right) \\ &= \left(\frac{3}{10} - \frac{1}{10} \right) + \left(\frac{1}{10} + \frac{3}{10} \right) i \\ &= \frac{2}{10} + \frac{4}{10} i \\ &= \frac{1}{5} + \frac{2}{5} i \end{aligned}$$

2. We proceed formally, using the multiplication by 1 trick. We have

$$\begin{aligned} \frac{w}{z} &= \frac{1+i}{3-i} \\ &= \frac{1+i}{3-i} \frac{3+i}{3+i} \\ &= \frac{3-1+(3+1)i}{3^2+1^2} \\ &= \frac{2}{10} + \frac{4}{10} i \\ &= \frac{1}{5} + \frac{2}{5} i \end{aligned}$$

Definition 4.9: Complex conjugate

Let $z = x + iy$. We call the **complex conjugate** of z, denoted by $\bar{z}$, the complex number

$$\bar{z} = x - iy\,.$$

Theorem 4.10

For all $z_1, z_2 \in \mathbb{C}$ it holds:

- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
- $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$

4.1 The complex plane

Definition 4.11: Modulus

The **modulus** of a complex number $z = x + iy$ is defined by

$$|z| := \sqrt{x^2 + y^2}\,.$$

Definition 4.12: Distance in $\mathbb{C}$

Given $z_1, z_2 \in \mathbb{C}$, we define the **distance** between z_1 and z_2 as the quantity

$$|z_1 - z_2| .$$

Theorem 4.13

Given $z_1, z_2 \in \mathbb{C}$, we have

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} .$$

Example 4.14

Question. Compute the distance between

$$z = 2 - 4i , \quad w = -5 + i .$$

Solution. The distance is

$$\begin{aligned} |z - w| &= |(2 - 4i) - (-5 + i)| \\ &= |7 - 5i| \\ &= \sqrt{7^2 + (-5)^2} \\ &= \sqrt{74} \end{aligned}$$

Theorem 4.15

Let $z, z_1, z_2 \in \mathbb{C}$. Then

1. $|z_1 \cdot z_2| = |z_1|\,|z_2|$
2. $|z^n| = |z|^n$ for all $n \in \mathbb{N}$
3. $z \cdot \bar{z} = |z|^2$

Theorem 4.16: Triangle inequality in $\mathbb{C}$

For all $x, y, z \in \mathbb{C}$,

1. $|x + y| \leq |x| + |y|$
2. $|x - z| \leq |x - y| + |y - z|$

Definition 4.17: Argument

Let $z \in \mathbb{C}$. The angle θ between the line connecting the origin and z and the positive real axis is called the **argument** of z, and is denoted by

$$\theta := \arg(z) .$$

Example 4.18

We have the following arguments:

$$\begin{aligned} &\arg(1) = 0 & &\arg(i) = \frac{\pi}{2} \\ &\arg(-1) = \pi & &\arg(-i) = -\frac{\pi}{2} \\ &\arg(1 + i) = \frac{1}{4}\pi & &\arg(-1 - i) = -\frac{3}{4}\pi \end{aligned}$$

Theorem 4.19: Polar coordinates

Let $z \in \mathbb{C}$ with $z = x + iy$ and $z \neq 0$. Then

$$x = \rho \cos(\theta) , \quad y = \rho \sin(\theta) ,$$

where

$$\rho := |z| = \sqrt{x^2 + y^2} , \quad \theta := \arg(z) .$$

Definition 4.20: Trigonometric form

Let $z \in \mathbb{C}$. The trigonometric form of z is

$$z = |z| \left[\cos(\theta) + i \sin(\theta) \right] ,$$

where $\theta = \arg(z)$.

Example 4.21

Question. Suppose that $z \in \mathbb{C}$ has polar coordinates

$$\rho = \sqrt{8} , \quad \theta = \frac{3}{4}\pi .$$

Therefore, the trigonometric form of z is

$$z = \sqrt{8} \left[\cos\left(\frac{3}{4}\pi\right) + i \sin\left(\frac{3}{4}\pi\right) \right] .$$

Write z in cartesian form.
Solution. We have

$$\begin{aligned} x &= \rho \cos(\theta) = \sqrt{8} \cos\left(\frac{3}{4}\pi\right) = -\sqrt{8} \cdot \frac{\sqrt{2}}{2} = -2 \\ y &= \rho \sin(\theta) = \sqrt{8} \sin\left(\frac{3}{4}\pi\right) = \sqrt{8} \cdot \frac{\sqrt{2}}{2} = 2 . \end{aligned}$$

Therefore, the cartesian form of z is

$$z = x + iy = -2 + 2i .$$

Corollary 4.22: Computing $\arg(z)$

Let $z \in \mathbb{C}$ with $z = x + iy$ and $z \neq 0$. Then

$$\arg(z) = \begin{cases} \arctan\left(\dfrac{y}{x}\right) & \text{if } x > 0 \\ \arctan\left(\dfrac{y}{x}\right) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\ \arctan\left(\dfrac{y}{x}\right) - \pi & \text{if } x < 0 \text{ and } y < 0 \\ \dfrac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\dfrac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \end{cases}$$

where arctan is the inverse of tan.

Example 4.23

Question. Compute the arguments of the complex numbers

$$z = 3 + 4i\,, \quad \bar{z} = 3 - 4i\,, \quad -\bar{z} = -3 + 4i\,, \quad -z = -3 - 4i\,.$$

Solution. Using the formula for arg in Corollary 4.22 we have

$$\begin{aligned} \arg(3+4i) &= \arctan\left(\frac{4}{3}\right) \\ \arg(3-4i) &= \arctan\left(-\frac{4}{3}\right) = -\arctan\left(\frac{4}{3}\right) \\ \arg(-3+4i) &= \arctan\left(-\frac{4}{3}\right) + \pi = -\arctan\left(\frac{4}{3}\right) + \pi \\ \arg(-3-4i) &= \arctan\left(\frac{4}{3}\right) - \pi \end{aligned}$$

Theorem 4.24: Euler's identity

For all $\theta \in \mathbb{R}$ it holds

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)\,.$$

Theorem 4.25

For all $\theta \in \mathbb{R}$ it holds

$$\left|e^{i\theta}\right| = 1\,.$$

Theorem 4.26

Let $z \in \mathbb{C}$ with $z = x + iy$ and $z \neq 0$. Then

$$z = \rho e^{i\theta}\,,$$

where

$$\rho := |z| = \sqrt{x^2 + y^2}\,, \qquad \theta := \arg(z)\,.$$

Definition 4.27: Exponential form

The **exponential form** of a complex number $z \in \mathbb{C}$ is

$$z = \rho e^{i\theta} = |z|\, e^{i\arg(z)}\,.$$

Example 4.28

Question. Write the number

$$z = -2 + 2i$$

in exponential form.
Solution. From Example 4.21 we know that $z = -2 + 2i$ can be written in trigonometric form as

$$z = \sqrt{8}\left[\cos\left(\frac{3}{4}\pi\right) + i\sin\left(\frac{3}{4}\pi\right)\right]\,.$$

By Euler's identity we hence obtain the exponential form

$$z = \sqrt{8}e^{i\frac{3}{4}\pi}\,.$$

Remark 4.29: Periodicity of exponential

For all $k \in \mathbb{Z}$ we have

$$e^{i\theta} = e^{i(\theta + 2\pi k)}\,, \tag{4.1}$$

meaning that the complex exponential is 2π-periodic.

Proposition 4.30

Let $z, z_1, z_2 \in \mathbb{C}$ and suppose that

$$z = \rho e^{i\theta}\,, \quad z_1 = \rho_1 e^{i\theta_1}\,, \quad z_2 = \rho_2 e^{i\theta_2}\,.$$

We have

$$z_1 \cdot z_2 = \rho_1\rho_2 e^{i(\theta_1 + \theta_2)}\,, \quad z^n = \rho^n e^{in\theta}\,,$$

for all $n \in \mathbb{N}$.

Example 4.31

Question. Compute $(-2 + 2i)^4$.
Solution. We have two possibilities:

1. Use the binomial theorem:

$$\begin{aligned}(-2+2i)^4 &= (-2)^4 + \binom{4}{1}(-2)^3 \cdot 2i + \binom{4}{2}(-2)^2 \cdot (2i)^2 \\ &\quad + \binom{4}{3}(-2)\cdot(2i)^3 + (2i)^4 \\ &= 16 - 4\cdot 8 \cdot 2i - 6 \cdot 4 \cdot 4 + 4 \cdot 2 \cdot 8i + 16 \\ &= 16 - 64i - 96 + 64i + 16 = -64\,.\end{aligned}$$

2. A much simpler calculation is possible by using the exponential form: We know that

$$-2+2i = \sqrt{8}e^{i\frac{3}{4}\pi}$$

by Example 4.28. Hence

$$(-2+2i)^4 = \left(\sqrt{8}e^{i\frac{3}{4}\pi}\right)^4 = 8^2 e^{i3\pi} = -64\,,$$

where we used that

$$e^{i3\pi} = e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$$

by 2π periodicity of $e^{i\theta}$ and Euler's identity.

Definition 4.32: Complex exponential

The complex exponential of $z \in \mathbb{C}$ is defined as

$$e^z = |z|e^{i\theta}\,, \quad \theta = \arg(z)\,.$$

Theorem 4.33

Let $z, w \in \mathbb{C}$. Then

$$e^{z+w} = e^z e^w\,, \quad (e^z)^w = e^{zw}\,. \tag{4.2}$$

Example 4.34

Question. Compute i^i.
Solution. We know that

$$|i| = 1\,, \quad \arg(i) = \frac{\pi}{2}\,.$$

Hence we can write i in exponential form

$$i = |i|e^{i\arg(i)} = e^{i\frac{\pi}{2}}\,.$$

Therefore

$$i^i = \left(e^{i\frac{\pi}{2}}\right)^i = e^{i^2\frac{\pi}{2}} = e^{-\frac{\pi}{2}}\,.$$

4.2 Fundamental Theorem of Algebra

Theorem 4.35: Fundamental theorem of algebra

Let $p_n(z)$ be a polynomial of degree n with complex coefficients, i.e.,

$$p_n(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0,$$

for some coefficients $a_n, \ldots, a_0 \in \mathbb{C}$ with $a_n \neq 0$. There exist

$$z_1, \ldots, z_n \in \mathbb{C}$$

solutions to the polynomial equation

$$p_n(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 = 0\,. \tag{4.3}$$

In particular, p_n factorizes as

$$p_n(z) = a_n(z-z_1)(z-z_2)\cdots(z-z_n)\,. \tag{4.4}$$

Example 4.36

Question. Find all the complex solutions to

$$z^2 = -1 \tag{4.5}$$

Solution. The equation $z^2 = -1$ is equivalent to

$$p(z) = 0\,, \quad p(z) := z^2 + 1\,.$$

Since p has degree $n = 2$, the Fundamental Theorem of Algebra tells us that there are two solutions to (4.5). We have already seen that these two solutions are $z = i$ and $z = -i$. Then p factorizes as

$$p(z) = z^2 + 1 = (z-i)(z+i)\,.$$

Example 4.37

Question. Find all the complex solutions to

$$z^4 - 1 = 0\,. \tag{4.6}$$

Solution The associated polynomial equation is

$$p(z) = 0\,, \quad p(z) := z^4 - 1\,.$$

Since p has degree $n = 4$, the Fundamental Theorem of Algebra tells us that there are 4 solutions to (4.6). Let us find such solutions. We use the well known identity

$$a^2 - b^2 = (a+b)(a-b)\,, \quad \forall\, a, b \in \mathbb{R}\,,$$

to factorize p. We get:

$$p(z) = (z^4 - 1) = (z^2+1)(z^2-1)\,.$$

We know that

$$z^2 + 1 = 0$$

has solutions $z = \pm i$. Instead

$$z^2 - 1 = 0$$

has solutions $x = \pm 1$. Hence, the four solutions of (4.6) are given by

$$z = 1, -1, i, -i\,,$$

and p factorizes as

$$p(z) = z^4 - 1 = (z-1)(z+1)(z-i)(z+i)\,.$$

Definition 4.38

Suppose that the polynomial p_n factorizes as

$$p_n(z) = a_n (z - z_1)^{k_1} (z - z_2)^{k_2} \cdots (z - z_m)^{k_m}$$

with $a_n \neq 0$, $z_1, \dots, z_m \in \mathbb{C}$ and $k_1, \dots, k_m \in \mathbb{N}$, $k_i \geq 1$. In this case p_n has degree

$$n = k_1 + \ldots + k_m = \sum_{i=1}^{m} k_i\,.$$

Note that z_i is solves the equation

$$p_n(z) = 0$$

exactly k_i times. We call k_i the **multiplicity** of the solution z_i.

Example 4.39

The equation

$$(z-1)(z-2)^2(z+i)^3 = 0$$

has 6 solutions:

- $z = 1$ with multiplicity 1
- $z = 2$ with multiplicity 2
- $z = -i$ with multiplicity 3

4.3 Solving polynomial equations

Proposition 4.40: Quadratic formula

Let $a, b, c \in \mathbb{R}, a \neq 0$ and consider the equation

$$ax^2 + bx + c = 0\,. \tag{4.7}$$

Define

$$\Delta := b^2 - 4ac \in \mathbb{R}\,.$$

The following hold:

1. If $\Delta > 0$ then (4.7) has two distinct real solutions $z_1, z_2 \in \mathbb{R}$ given by
$$z_1 = \frac{-b - \sqrt{\Delta}}{2a}\,, \quad z_2 = \frac{-b + \sqrt{\Delta}}{2a}\,.$$
2. If $\Delta = 0$ then (4.7) has one real solution $z \in \mathbb{R}$ with multiplicity 2. Such solution is given by
$$z = z_1 = z_2 = \frac{-b}{2a}\,.$$
3. If $\Delta < 0$ then (4.7) has two distinct complex solutions $z_1, z_2 \in \mathbb{C}$ given by
$$z_1 = \frac{-b - i\sqrt{-\Delta}}{2a}\,, \quad z_2 = \frac{-b + i\sqrt{-\Delta}}{2a}\,,$$
where $\sqrt{-\Delta} \in \mathbb{R}$, since $-\Delta > 0$.

In all cases, the polynomial at (4.7) factorizes as

$$az^2 + bz + c = a(z - z_1)(z - z_2)\,.$$

Example 4.41

Question. Solve the following equations:

1. $3z^2 - 6z + 2 = 0$
2. $4z^2 - 8z + 4 = 0$
3. $z^2 + 2z + 3 = 0$

Solution.

1. We have that
$$\Delta = (-6)^2 - 4 \cdot 3 \cdot 2 = 12 > 0$$
Therefore the equation has two distinct real solutions, given by
$$z = \frac{-(-6) \pm \sqrt{12}}{2 \cdot 3} = \frac{6 \pm \sqrt{12}}{6} = 1 \pm \frac{\sqrt{3}}{3}$$
In particular we have the factorization
$$3z^2 - 6z + 2 = 3\left(z - 1 - \frac{\sqrt{3}}{3}\right)\left(z - 1 + \frac{\sqrt{3}}{3}\right)\,.$$

2. We have that

$$\Delta = (-8)^2 - 4 \cdot 4 \cdot 4 = 0\,.$$

Therefore there exists one solution with multiplicity 2. This is given by

$$z = \frac{-(-8)}{2 \cdot 4} = 1\,.$$

In particular we have the factorization

$$4z^2 - 8x + 4 = 4(z-1)^2\,.$$

3. We have

$$\Delta = 2^2 - 4 \cdot 1 \cdot 3 = -8 < 0\,.$$

Therefore there are two complex solutions given by

$$z = \frac{-2 \pm i\sqrt{8}}{2 \cdot 1} = -1 \pm i\sqrt{2}\,.$$

In particular we have the factorization

$$z^2 + 2z + 3 = (z + 1 - i\sqrt{2})(z + 1 + i\sqrt{2})\,.$$

Proposition 4.42: Quadratic formula with complex coefficients

Let $a, b, c \in \mathbb{C}, a \neq 0$. The two solutions to

$$az^2 + bz + c = 0$$

are given by

$$z_1 = \frac{-b + S_1}{2a}\,, \quad z_2 = \frac{-b + S_2}{2a}\,,$$

where S_1 and S_2 are the two solutions to

$$z^2 = \Delta\,, \quad \Delta := b^2 - 4ac\,.$$

Example 4.43

Question Find all the solutions to

$$\frac{1}{2}z^2 - (3+i)z + (4-i) = 0\,. \tag{4.8}$$

Solution. We have

$$\begin{aligned} \Delta &= (-(3+i))^2 - 4 \cdot \frac{1}{2} \cdot (4-i) \\ &= 8 + 6i - 8 + 2i \\ &= 8i\,. \end{aligned}$$

Therefore $\Delta \in \mathbb{C}$. We have to find solutions S_1 and S_2 to the equation

$$z^2 = \Delta = 8i\,. \tag{4.9}$$

We look for solutions of the form $z = a + ib$. Then we must have that

$$z^2 = (a+ib)^2 = a^2 - b^2 + 2abi = 8i\,.$$

Thus

$$a^2 - b^2 = 0\,, \quad 2ab = 8\,.$$

From the first equation we conclude that $|a| = |b|$. From the second equation we have that $ab = 4$, and therefore a and b must have the same sign. Hence $a = b$, and

$$2ab = 8 \quad \Longrightarrow \quad a = b = \pm 2\,.$$

From this we conclude that the solutions to (4.9) are

$$S_1 = 2 + 2i\,, \quad S_2 = -2 - 2i\,.$$

Hence the solutions to (4.8) are

$$\begin{aligned} z_1 &= \frac{3 + i + S_1}{2 \cdot \frac{1}{2}} = 3 + i + S_1 \\ &= 3 + i + 2 + 2i = 5 + 3i\,, \end{aligned}$$

and

$$\begin{aligned} z_2 &= \frac{3 + i + S_2}{2 \cdot \frac{1}{2}} = 3 + i + S_2 \\ &= 3 + i - 2 - 2i = 1 - i\,. \end{aligned}$$

In particular, the given polynomial factorizes as

$$\begin{aligned} \frac{1}{2}z^2 - (3+i)z + (4-i) &= \frac{1}{2}(z - z_1)(z - z_2) \\ &= \frac{1}{2}(z - 5 - 3i)(z - 1 + i)\,. \end{aligned}$$

Example 4.44

Question. Consider the equation

$$z^3 - 7z^2 + 6z = 0\,.$$

1. Check whether $z = 0, 1, -1$ are solutions.
2. Using your answer from Point 1, and polynomial division, find all the solutions.

Solution.

1. By direct inspection we see that $z = 0$ and $z = 1$ are solutions.

2. Since $z = 0$ is a solution, we can factorize

$$z^3 - 7z^2 + 6z = z\left(z^2 - 7z + 6\right) .$$

We could now use the quadratic formula on the term $z^2 - 7z + 6$ to find the remaining two roots. However, we have already observed that $z = 1$ is a solution. Therefore $z - 1$ divides $z^2 - 7z + 6$. Using polynomial long division, we find that

$$\frac{z^2 - 7z + 6}{z - 1} = z - 6 .$$

Therefore the last solution is $z = 6$, and

Example 4.45

Question. Find all the complex solutions to

$$z^3 - 7z + 6 = 0 .$$

Solution. It is easy to see $z = 1$ is a solution. This means that $z - 1$ divides $z^3 - 7z + 6$. By using polynomial long division, we compute that

$$\frac{z^3 - 7z + 6}{z - 1} = z^2 + z - 6 .$$

We are now left to solve

$$z^2 + z - 6 = 0 .$$

Using the quadratic formula, we see that the above is solved by $z = 2$ and $z = -3$. Therefore the given polynomial factorizes as

$$z^3 - 7z + 6 = (z - 1)(z - 2)(z + 3) .$$

4.4 Roots

Theorem 4.46

Let $n \in \mathbb{N}$ and consider the equation

$$z^n = 1 . \tag{4.10}$$

All the n solutions to (4.10) are given by

$$z_k = \exp\left(i\frac{2\pi k}{n}\right) , \quad k = 0, \dots, n-1 ,$$

where $\exp(x)$ denotes e^x.

Definition 4.47

The n solutions to

$$z^n = 1$$

are called the **roots of unity**.

Example 4.48

Question. Find all the solutions to

$$z^4 = 1 .$$

Solution. The 4 solutions are given by

$$z_k = \exp\left(i\frac{2\pi k}{4}\right) = \exp\left(i\frac{\pi k}{2}\right) ,$$

for $k = 0, 1, 2, 3$. We compute:

$$z_0 = e^{i0} = 1 , \qquad z_1 = e^{i\frac{\pi}{2}} = i ,$$
$$z_2 = e^{i\pi} = -1 , \qquad z_3 = e^{i\frac{3\pi}{2}} = -i .$$

Note that for $k = 4$ we would again get the solution $z = e^{i2\pi} = 1$.

Example 4.49

Question. Find all the solutions to

$$z^3 = 1 .$$

Solution. The 3 solutions are given by

$$z_k = \exp\left(i\frac{2\pi k}{3}\right) ,$$

for $k = 0, 1, 2$. We compute:

$$z_0 = e^{i0} = 1, \quad z_1 = e^{i\frac{2\pi}{3}}, \quad z_2 = e^{i\frac{4\pi}{3}} .$$

We can write z_1 and z_2 in cartesian form:

$$z_1 = e^{i\frac{2\pi}{3}} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

and

$$z_2 = e^{i\frac{4\pi}{3}} = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i\,.$$

Theorem 4.50

Let $n \in \mathbb{N}$, $c \in \mathbb{C}$ and consider the equation

$$z^n = c\,. \tag{4.11}$$

All the n solutions to (4.11) are given by

$$z_k = \sqrt[n]{|c|}\,\exp\left(i\,\frac{\theta + 2\pi k}{n}\right)\,, \qquad k = 0, \dots, n-1\,,$$

where $\sqrt[n]{|c|}$ is the n-th root of the real number $|c|$, and $\theta = \arg(c)$.

Example 4.51

Question. Find all the $z \in \mathbb{C}$ such that

$$z^5 = -32\,.$$

Solution. Let $c = -32$. We have

$$|c| = |-32| = 32 = 2^5\,, \qquad \theta = \arg(-32) = \pi\,.$$

The 5 solutions are given by

$$z_k = (2^5)^{\frac{1}{5}}\exp\left(i\pi\,\frac{1+2k}{5}\right)\,, \qquad k \in \mathbb{Z}\,,$$

for $k = 0, 1, 2, 3, 4$. We get

$$\begin{aligned} z_0 &= 2e^{i\frac{\pi}{5}} & z_1 &= 2e^{i\frac{3\pi}{5}} \\ z_2 &= 2e^{i\pi} = -2 & z_3 &= 2e^{i\frac{7\pi}{5}} \\ z_4 &= 2e^{i\frac{9\pi}{5}} \end{aligned}$$

Example 4.52

Question. Find all the $z \in \mathbb{C}$ such that

$$z^4 = 9\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)\,.$$

Solution. Set

$$c := 9\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)\,.$$

The complex number c is already in the trigonometric form, so that we can immediately obtain

$$|c| = 9\,, \qquad \theta = \arg(c) = \frac{\pi}{3}\,.$$

The 4 solutions are given by

$$\begin{aligned} z_k &= \sqrt[4]{9}\,\exp\left(i\,\frac{\pi/3 + 2\pi k}{4}\right) \\ &= \sqrt{3}\exp\left(i\pi\,\frac{1+6k}{12}\right) \end{aligned}$$

for $k = 0, 1, 2, 3$. We compute

$$\begin{aligned} z_0 &= \sqrt{3}e^{i\pi\frac{1}{12}} & z_1 &= \sqrt{3}e^{i\pi\frac{7}{12}} \\ z_2 &= \sqrt{3}e^{i\pi\frac{13}{12}} & z_3 &= \sqrt{3}e^{i\pi\frac{19}{12}} \end{aligned}$$

5 Sequences in $\mathbb{R}$

Definition 5.1: Convergent sequence

The real sequence (a_n) **converges** to a, or equivalently has limit a, denoted by

$$\lim_{n\to\infty} a_n = a\,,$$

if for all $\varepsilon \in \mathbb{R}, \varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}, n \geq N$ it holds that

$$|a_n - a| < \varepsilon\,.$$

Using quantifiers, we can write this as

$$\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N} \ \text{ s.t. } \ \forall\, n \geq N\,,\ |a_n - a| < \varepsilon\,.$$

The sequence $(a_n)_{n\in\mathbb{N}}$ is **convergent** if it admits limit.

Theorem 5.2

Let $p > 0$. Then

$$\lim_{n\to\infty} \frac{1}{n^p} = 0\,.$$

Proof

Let $p > 0$. We have to show that

$$\forall \varepsilon > 0\,,\ \exists\, N \in \mathbb{N} \ \text{ s.t. } \ \forall\, n \geq N\,,\ \left|\frac{1}{n^p} - 0\right| < \varepsilon\,.$$

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$N > \frac{1}{\varepsilon^{1/p}}\,. \tag{5.1}$$

Let $n \geq N$. Since $p > 0$, we have $n^p \geq N^p$, which implies

$$\frac{1}{n^p} \leq \frac{1}{N^p}\,.$$

By (5.1) we deduce

$$\frac{1}{N^p} < \varepsilon\,.$$

Then

$$\left|\frac{1}{n^p} - 0\right| = \frac{1}{n^p} \leq \frac{1}{N^p} < \varepsilon\,.$$

Example 5.3

Question. Using the definition of convergence, prove that

$$\lim_{n\to\infty} \frac{n}{2n+3} = \frac{1}{2}\,.$$

Solution.

1. *Rough Work:* Let $\varepsilon > 0$. We want to find $N \in \mathbb{N}$ such that

   $$\left|\frac{n}{2n+3} - \frac{1}{2}\right| < \varepsilon\,, \quad \forall\, n \geq N\,.$$

   To this end, we compute:

   $$\begin{aligned}\left|\frac{n}{2n+3} - \frac{1}{2}\right| &= \left|\frac{2n-(2n+3)}{2(2n+3)}\right| \\ &= \left|\frac{-3}{4n+6}\right| \\ &= \frac{3}{4n+6}\,.\end{aligned}$$

   Therefore

   $$\begin{aligned}\left|\frac{n}{2n+3} - \frac{1}{2}\right| < \varepsilon \quad &\iff \quad \frac{3}{4n+6} < \varepsilon \\ &\iff \quad \frac{4n+6}{3} > \frac{1}{\varepsilon} \\ &\iff \quad 4n+6 > \frac{3}{\varepsilon} \\ &\iff \quad 4n > \frac{3}{\varepsilon} - 6 \\ &\iff \quad n > \frac{3}{4\varepsilon} - \frac{6}{4}\,.\end{aligned}$$

   Looking at the above equivalences, it is clear that $N \in \mathbb{N}$ has to be chosen so that

   $$N > \frac{3}{4\varepsilon} - \frac{6}{4}\,.$$

2. *Formal Proof:* We have to show that

   $$\forall \varepsilon > 0\,,\ \exists\, N \in \mathbb{N} \ \text{ s.t. } \ \forall\, n \geq N\,,\ \left|\frac{n}{2n+3} - \frac{1}{2}\right| < \varepsilon\,.$$

   Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that

   $$N > \frac{3}{4\varepsilon} - \frac{6}{4}\,. \tag{5.2}$$

   By the rough work shown above, inequality (5.2) is equivalent to

   $$\frac{3}{4N+6} < \varepsilon\,.$$

Let $n \geq N$. Then

$$\begin{aligned} \left|\frac{n}{2n+3} - \frac{1}{2}\right| &= \frac{3}{4n+6} \\ &\leq \frac{3}{4N+6} \\ &< \varepsilon \,, \end{aligned}$$

where in the third line we used that $n \geq N$.

Definition 5.4: Divergent sequence

We say that a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ is **divergent** if it is not convergent.

Theorem 5.5

Let (a_n) be the sequence defined by

$$a_n = (-1)^n \,.$$

Then (a_n) does not converge.

Proof

Suppose by contradiction that $a_n \to a$ for some $a \in \mathbb{R}$. Let

$$\varepsilon := \frac{1}{2} \,.$$

Since $a_n \to a$, there exists $N \in \mathbb{N}$ such that

$$|a_n - a| < \varepsilon = \frac{1}{3} \quad \forall \, n \geq N \,.$$

If we take $n = 2N$, then $n \geq N$ and

$$|a_{2N} - a| = |1 - a| < \frac{1}{2} \,.$$

If we take $n = 2N + 1$, then $n \geq N$ and

$$|a_{2N+1} - a| = |-1 - a| < \frac{1}{2} \,.$$

Therefore

$$\begin{aligned} 2 &= |(1-a) - (-1-a)| \\ &\leq |1-a| + |-1-a| \\ &< \frac{1}{2} + \frac{1}{2} = 1 \,, \end{aligned}$$

which is a contradiction. Hence (a_n) is divergent.

Theorem 5.6: Uniqueness of limit

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence. Suppose that

$$\lim_{n\to\infty} a_n = a \,, \quad \lim_{n\to\infty} a_n = b \,.$$

Then $a = b$.

Definition 5.7: Bounded sequence

A sequence $(a_n)_{n\in\mathbb{N}}$ is called **bounded** if there exists a constant $M \in \mathbb{R}$, with $M > 0$, such that

$$|a_n| \leq M \,, \quad \forall \, n \in \mathbb{N} \,.$$

Theorem 5.8

Every convergent sequence is bounded.

Example 5.9

The sequence

$$a_n = (-1)^n$$

is bounded but not convergent.

Corollary 5.10

If a sequence is not bounded, then the sequence does not converge.

Remark 5.11

For a sequence (a_n) to be unbounded, it means that

$$\forall \, M > 0 \,, \; \exists \, n \in \mathbb{N} \;\text{ s.t. }\; |a_n| > M \,.$$

Theorem 5.12

For all $p > 0$, the sequence

$$a_n = n^p$$

does not converge.

Theorem 5.13

The sequence

$$a_n = \log n$$

does not converge.

Theorem 5.14: Algebra of limits

Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be sequences in $\mathbb{R}$. Suppose that

$$\lim_{n\to\infty} a_n = a\,, \quad \lim_{n\to\infty} b_n = b\,,$$

for some $a, b \in \mathbb{R}$. Then,

1. Limit of sum is the sum of limits:
$$\lim_{n\to\infty} (a_n \pm b_n) = a \pm b$$
2. Limit of product is the product of limits:
$$\lim_{n\to\infty} (a_n b_n) = ab$$
3. If $b_n \neq 0$ for all $n \in \mathbb{N}$ and $b \neq 0$, then
$$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{a}{b}$$

Example 5.15

Question. Prove that

$$\lim_{n\to\infty} \frac{3n}{7n+4} = \frac{3}{7}\,.$$

Solution. We can rewrite

$$\frac{3n}{7n+4} = \frac{3}{7+\frac{4}{n}}$$

From Theorem 5.2, we know that

$$\frac{1}{n} \to 0\,.$$

Hence, it follows from Theorem 5.14 Point 2 that

$$\frac{4}{n} = 4 \cdot \frac{1}{n} \to 4 \cdot 0 = 0\,.$$

By Theorem 5.14 Point 1 we have

$$7 + \frac{4}{n} \to 7 + 0 = 7\,.$$

Finally we can use Theorem 5.14 Point 3 to infer

$$\frac{3n}{7n+4} = \frac{3}{7+\frac{4}{n}} \to \frac{3}{7}\,.$$

Example 5.16

Question. Prove that

$$\lim_{n\to\infty} \frac{n^2-1}{2n^2-3} = \frac{1}{2}\,.$$

Solution. Factor n^2 to obtain

$$\frac{n^2-1}{2n^2-3} = \frac{1-\frac{1}{n^2}}{2-\frac{3}{n^2}}\,.$$

By Theorem 5.2 we have

$$\frac{1}{n^2} \to 0\,.$$

We can then use the Algebra of Limits Theorem 5.14 Point 2 to infer

$$\frac{3}{n^2} \to 3 \cdot 0 = 0$$

and Theorem 5.14 Point 1 to get

$$1 - \frac{1}{n^2} \to 1 - 0 = 1\,, \quad 2 - \frac{3}{n^2} \to 2 - 0 = 2\,.$$

Finally we use Theorem 5.14 Point 3 and conclude

$$\frac{1-\frac{1}{n^2}}{2-\frac{3}{n^2}} \to \frac{1}{2}\,.$$

Therefore

$$\lim_{n\to\infty} \frac{n^2-1}{2n^2-3} = \lim_{n\to\infty} \frac{1-\frac{1}{n^2}}{2-\frac{3}{n^2}} = \frac{1}{2}\,.$$

Example 5.17

Question. Prove that the sequence

$$a_n = \frac{4n^3+8n+1}{7n^2+2n+1}$$

does not converge.

Solution. To show that the sequence (a_n) does not converge, we divide by the largest power in the denominator, which in this case is n^2

$$a_n = \frac{4n^3 + 8n + 1}{7n^2 + 2n + 1} = \frac{4n + \frac{8}{n} + \frac{1}{n^2}}{7 + \frac{2}{n} + \frac{1}{n^2}} = \frac{b_n}{c_n}$$

where we set

$$b_n := 4n + \frac{8}{n} + \frac{1}{n^2}\,, \quad c_n := 7 + \frac{2}{n} + \frac{1}{n^2}\,.$$

Using the Algebra of Limits Theorem 5.14 we see that

$$c_n = 7 + \frac{2}{n} + \frac{1}{n^2} \to 7\,.$$

Suppose by contradiction that

$$a_n \to a$$

for some $a \in \mathbb{R}$. Then, by the Algebra of Limits, we would infer

$$b_n = c_n \cdot a_n \to 7a\,,$$

concluding that b_n is convergent to $7a$. We have that

$$b_n = 4n + d_n\,, \quad d_n := \frac{8}{n} + \frac{1}{n^2}\,.$$

Again by the Algebra of Limits Theorem 5.14 we get that

$$d_n = \frac{8}{n} + \frac{1}{n^2} \to 0\,,$$

and hence

$$4n = b_n - d_n \to 7a - 0 = 7a\,.$$

This is a contradiction, since the sequence $(4n)$ is unbounded, and hence cannot be convergent. Hence (a_n) is not convergent.

Example 5.18

Question. Define the sequence

$$a_n := \frac{2n^3 + 7n + 1}{5n + 9} \cdot \frac{8n + 9}{6n^3 + 8n^2 + 3}\,.$$

Prove that

$$\lim_{n\to\infty} a_n = \frac{8}{15}\,.$$

Solution. The first fraction in (a_n) does not converge, as it is unbounded. Therefore we cannot use Point 2 in Theorem 5.14 directly. However, we note that

$$a_n = \frac{2n^3 + 7n + 1}{5n + 9} \cdot \frac{8n + 9}{6n^3 + 8n^2 + 3} = \frac{8n + 9}{5n + 9} \cdot \frac{2n^3 + 7n + 1}{6n^3 + 8n^2 + 3}\,.$$

Factoring out n and n^3, respectively, and using the Algebra of Limits, we see that

$$\frac{8n + 9}{5n + 9} = \frac{8 + 9/n}{5 + 9/n} \to \frac{8 + 0}{5 + 0} = \frac{8}{5}$$

and

$$\frac{2 + 7/n^2 + 1/n^3}{6 + 8/n + 3/n^3} \to \frac{2 + 0 + 0}{6 + 0 + 0} = \frac{1}{3}$$

Therefore Theorem 5.14 Point 2 ensures that

$$a_n \to \frac{8}{5} \cdot \frac{1}{3} = \frac{8}{15}\,.$$

Example 5.19

Question. Prove that

$$a_n = \frac{n^{7/3} + 2\sqrt{n} + 7}{4n^{3/2} + 5n}$$

does not converge.

Solution. The largest power of n in the denominator is $n^{3/2}$. Hence we factor out $n^{3/2}$

$$a_n = \frac{n^{7/3} + 2\sqrt{n} + 7}{4n^{3/2} + 5n} = \frac{n^{7/3-3/2} + 2n^{1/2-3/2} + 7n^{-3/2}}{4 + 5n^{-3/2}} = \frac{n^{5/6} + 2n^{-1} + 7n^{-3/2}}{4 + 5n^{-3/2}} = \frac{b_n}{c_n}$$

where we set

$$b_n := n^{5/6} + 2n^{-1} + 7n^{-3/2}\,, \quad c_n := 4 + 5n^{-3/2}\,.$$

We see that b_n is unbounded while $c_n \to 4$. By the Algebra of Limits (and usual contradiction argument) we conclude that (a_n) is divergent.

Theorem 5.20

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ such that

$$\lim_{n\to\infty} a_n = a\,,$$

for some $a \in \mathbb{R}$. If $a_n \geq 0$ for all $n \in \mathbb{N}$ and $a \geq 0$, then

$$\lim_{n\to\infty} \sqrt{a_n} = \sqrt{a}\,.$$

Example 5.21

Question. Define the sequence

$$a_n = \sqrt{9n^2+3n+1} - 3n\,.$$

Prove that

$$\lim_{n\to\infty} a_n = \frac{1}{2}\,.$$

Solution. We first rewrite

$$\begin{aligned} a_n &= \sqrt{9n^2+3n+1} - 3n \\ &= \frac{\left(\sqrt{9n^2+3n+1} - 3n\right)\left(\sqrt{9n^2+3n+1} + 3n\right)}{\sqrt{9n^2+3n+1}+3n} \\ &= \frac{9n^2+3n+1-(3n)^2}{\sqrt{9n^2+3n+1}+3n} \\ &= \frac{3n+1}{\sqrt{9n^2+3n+1}+3n}\,. \end{aligned}$$

The biggest power of n in the denominator is n. Therefore we factor out n:

$$\begin{aligned} a_n &= \sqrt{9n^2+3n+1} - 3n \\ &= \frac{3n+1}{\sqrt{9n^2+3n+1}+3n} \\ &= \frac{3+\frac{1}{n}}{\sqrt{9+\frac{3}{n}+\frac{1}{n^2}}+3}\,. \end{aligned}$$

By the Algebra of Limits we have

$$9 + \frac{3}{n} + \frac{1}{n^2} \to 9+0+0 = 9\,.$$

Therefore we can use Theorem 5.20 to infer

$$\sqrt{9+\frac{3}{n}+\frac{1}{n^2}} \to \sqrt{9}\,.$$

By the Algebra of Limits we conclude:

$$a_n = \frac{3+\frac{1}{n}}{\sqrt{9+\frac{3}{n}+\frac{1}{n^2}}+3} \to \frac{3+0}{\sqrt{9}+3} = \frac{1}{2}\,.$$

Example 5.22

Question. Prove that the sequence

$$a_n = \sqrt{9n^2+3n+1} - 2n$$

does not converge.

Solution. We rewrite a_n as

$$\begin{aligned} a_n &= \sqrt{9n^2+3n+1} - 2n \\ &= \frac{(\sqrt{9n^2+3n+1} - 2n)(\sqrt{9n^2+3n+1} + 2n)}{\sqrt{9n^2+3n+1}+2n} \\ &= \frac{9n^2+3n+1-(2n)^2}{\sqrt{9n^2+3n+1}+2n} \\ &= \frac{5n^2+3n+1}{\sqrt{9n^2+3n+1}+2n} \\ &= \frac{5n+3+\frac{1}{n}}{\sqrt{9+\frac{3}{n}+\frac{1}{n^2}}+2} \\ &= \frac{b_n}{c_n}\,, \end{aligned}$$

where we factored n, being it the largest power of n in the denominator, and defined

$$b_n := 5n+3+\frac{1}{n}\,, \quad c_n := \sqrt{9+\frac{3}{n}+\frac{1}{n^2}}+2\,.$$

Note that

$$9+\frac{3}{n}+\frac{1}{n^2} \to 9$$

by the Algebra of Limits. Therefore

$$\sqrt{9+\frac{3}{n}+\frac{1}{n^2}} \to \sqrt{9} = 3$$

by Theorem 5.20. Hence

$$c_n = \sqrt{9+\frac{3}{n}+\frac{1}{n^2}}+2 \to 3+2 = 5\,.$$

The numerator

$$b_n = 5n+3+\frac{1}{n}$$

is instead unbounded. Therefore (a_n) is not convergent, by the Algebra of Limits and the usual contradiction argument.

5.1 Limit Tests

Theorem 5.23: Squeeze theorem

Let (a_n), (b_n) and (c_n) be sequences in $\mathbb{R}$. Suppose that

$$b_n \le a_n \le c_n \,, \quad \forall\, n \in \mathbb{N} \,,$$

and that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L \,.$$

Then

$$\lim_{n\to\infty} a_n = L \,.$$

Example 5.24

Question. Prove that

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0 \,.$$

Solution. For all $n \in \mathbb{N}$ we can estimate

$$-1 \le (-1)^n \le 1 \,.$$

Therefore

$$\frac{-1}{n} \le \frac{(-1)^n}{n} \le \frac{1}{n} \,, \quad \forall\, n \in \mathbb{N} \,.$$

Moreover

$$\lim_{n\to\infty} \frac{-1}{n} = -1 \cdot 0 = 0 \,, \quad \lim_{n\to\infty} \frac{1}{n} = 0 \,.$$

By the Squeeze Theorem 5.23 we conclude

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0 \,.$$

Example 5.25

Question. Prove that

$$\lim_{n\to\infty} \frac{\cos(3n) + 9n^2}{11n^2 + 15\sin(17n)} = \frac{9}{11} \,.$$

Solution. We know that

$$-1 \le \cos(x) \le 1 \,, \quad -1 \le \sin(x) \le 1 \,, \quad \forall\, x \in \mathbb{R} \,.$$

Therefore, for all $n \in \mathbb{N}$

$$-1 \le \cos(3n) \le 1 \,, \quad -1 \le \sin(17n) \le 1 \,.$$

We can use the above to estimate the numerator in the given sequence:

$$-1 + 9n^2 \le \cos(3n) + 9n^2 \le 1 + 9n^2 \,. \tag{5.3}$$

Concerning the denominator, we have

$$11n^2 - 15 \le 11n^2 + 15\sin(17n) \le 11n^2 + 15$$

and therefore

$$\frac{1}{11n^2 + 15} \le \frac{1}{11n^2 + 15\sin(17n)} \le \frac{1}{11n^2 - 15} \,. \tag{5.4}$$

Putting together (5.3)-(5.4) we obtain

$$\frac{-1 + 9n^2}{11n^2 + 15} \le \frac{\cos(3n) + 9n^2}{11n^2 + 15\sin(17n)} \le \frac{1 + 9n^2}{11n^2 - 15} \,.$$

By the Algebra of Limits we infer

$$\frac{-1 + 9n^2}{11n^2 + 15} = \frac{-\frac{1}{n^2} + 9}{11 + \frac{15}{n^2}} \to \frac{0 + 9}{11 + 0} = \frac{9}{11}$$

and

$$\frac{1 + 9n^2}{11n^2 - 15} = \frac{\frac{1}{n^2} + 9}{11 - \frac{15}{n^2}} \to \frac{0 + 9}{11 + 0} = \frac{9}{11} \,.$$

Applying the Squeeze Theorem 5.23 we conclude

$$\lim_{n\to\infty} \frac{\cos(3n) + 9n^2}{11n^2 + 15\sin(17n)} = \frac{9}{11} \,.$$

Theorem 5.26: Geometric Sequence Test

Let $x \in \mathbb{R}$ and let (a_n) be the geometric sequence defined by

$$a_n := x^n \,.$$

We have:

1. If $|x| < 1$, then
$$\lim_{n\to\infty} a_n = 0 \,.$$
2. If $|x| > 1$, then sequence (a_n) is unbounded, and hence divergent.

Example 5.27

We can apply Theorem 5.26 to prove convergence or divergence for the following sequences.

1. We have
$$\left(\frac{1}{2}\right)^n \longrightarrow 0$$
since
$$\left|\frac{1}{2}\right| = \frac{1}{2} < 1 \,.$$

2. We have
$$\left(\frac{-1}{2}\right)^n \longrightarrow 0$$
since
$$\left|\frac{-1}{2}\right| = \frac{1}{2} < 1\,.$$
3. The sequence
$$a_n = \left(\frac{-3}{2}\right)^n$$
does not converge, since
$$\left|\frac{-3}{2}\right| = \frac{3}{2} > 1\,.$$
4. As $n \to \infty$,
$$\frac{3^n}{(-5)^n} = \left(-\frac{3}{5}\right)^n \longrightarrow 0$$
since
$$\left|-\frac{3}{5}\right| = \frac{3}{5} < 1\,.$$
5. The sequence
$$a_n = \frac{(-7)^n}{2^{2n}}$$
does not converge, since
$$\frac{(-7)^n}{2^{2n}} = \frac{(-7)^n}{(2^2)^n} = \left(-\frac{7}{4}\right)^n$$
and
$$\left|-\frac{7}{4}\right| = \frac{7}{4} > 1\,.$$

Theorem 5.28: Ratio Test

Let (a_n) be a sequence in $\mathbb{R}$ such that
$$a_n \neq 0\,, \quad \forall\, n \in \mathbb{N}\,.$$

1. Suppose that the following limit exists:
$$L := \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|\,.$$
Then,
   - If $L < 1$ we have
   $$\lim_{n\to\infty} a_n = 0\,.$$
   - If $L > 1$, the sequence (a_n) is unbounded, and hence does not converge.

2. Suppose that there exists $N \in \mathbb{N}$ and $L > 1$ such that
$$\left|\frac{a_{n+1}}{a_n}\right| \geq L\,, \quad \forall\, n \geq N\,.$$
Then the sequence (a_n) is unbounded, and hence does not converge.

Example 5.29

Question. Let
$$a_n = \frac{3^n}{n!}\,,$$
where we recall that $n!$ (pronounced n factorial) is defined by
$$n! := n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1\,.$$
Prove that
$$\lim_{n\to\infty} a_n = 0\,.$$
Solution. We have
$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{\left(\frac{3^{n+1}}{(n+1)!}\right)}{\left(\frac{3^n}{n!}\right)} \\ &= \frac{3^{n+1}}{3^n}\,\frac{n!}{(n+1)!} \\ &= \frac{3\cdot 3^n}{3^n}\,\frac{n!}{(n+1)n!} \\ &= \frac{3}{n+1} \longrightarrow L = 0\,.\end{aligned}$$
Hence, $L = 0 < 1$ so $a_n \to 0$ by the Ratio Test in Theorem 5.28.

Example 5.30

Question. Consider the sequence
$$a_n = \frac{n! \cdot 3^n}{\sqrt{(2n)!}}\,.$$
Prove that (a_n) is divergent.
Solution. We have
$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{(n+1)! \cdot 3^{n+1}}{\sqrt{(2(n+1))!}}\,\frac{\sqrt{(2n)!}}{n! \cdot 3^n} \\ &= \frac{(n+1)!}{n!} \cdot \frac{3^{n+1}}{3^n} \cdot \frac{\sqrt{(2n)!}}{\sqrt{(2(n+1))!}}\end{aligned}$$
For the first two fractions we have
$$\frac{(n+1)!}{n!} \cdot \frac{3^{n+1}}{3^n} = 3(n+1)\,,$$

while for the third fraction

$$\frac{\sqrt{(2n)!}}{\sqrt{(2(n+1))!}} = \sqrt{\frac{(2n)!}{(2n+2)!}} = \sqrt{\frac{(2n)!}{(2n+2)\cdot(2n+1)\cdot(2n)!}} = \frac{1}{\sqrt{(2n+1)(2n+2)}} .$$

Therefore, using the Algebra of Limits,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3(n+1)}{\sqrt{(2n+1)(2n+2)}} = \frac{3n\left(1+\frac{1}{n}\right)}{\sqrt{n^2\left(2+\frac{1}{n}\right)\left(2+\frac{2}{n}\right)}} = \frac{3\left(1+\frac{1}{n}\right)}{\sqrt{\left(2+\frac{1}{n}\right)\left(2+\frac{2}{n}\right)}} \longrightarrow \frac{3}{\sqrt{4}} = \frac{3}{2} > 1 .$$

By the Ratio Test we conclude that (a_n) is divergent.

Example 5.31

Question. Prove that the following sequence is divergent

$$a_n = \frac{n!}{100^n} .$$

Solution. We have

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{100^n}{100^{n+1}} \frac{(n+1)!}{n!} = \frac{n+1}{100} .$$

Choose $N = 101$. Then for all $n \geq N$,

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{100} \geq \frac{N+1}{100} = \frac{101}{100} > 1 .$$

Hence a_n is divergent by the Ratio Test.

Let (a_n) be a real sequence. We say that:

1. (a_n) is **increasing** if
$$a_n \leq a_{n+1} , \quad \forall\, n \geq N .$$
2. (a_n) is **decreasing** if
$$a_n \geq a_{n+1} , \quad \forall\, n \geq N .$$
3. (a_n) is **monotone** if it is either increasing or decreasing.

Example 5.33

Question. Prove that the following sequence is increasing

$$a_n = \frac{n-1}{n} .$$

Solution. We have

$$a_{n+1} = \frac{n}{n+1} > \frac{n-1}{n} = a_n ,$$

where the inequality holds because

$$\frac{n}{n+1} > \frac{n-1}{n} \iff n^2 > (n-1)(n+1) \iff n^2 > n^2 - 1 \iff 0 > -1$$

Example 5.34

Question. Prove that the following sequence is decreasing

$$a_n = \frac{1}{n} .$$

Solution. We have

$$a_n = \frac{1}{n} > \frac{1}{n+1} = a_{n+1} ,$$

concluding.

Theorem 5.35: Monotone Convergence Theorem

Let (a_n) be a sequence in $\mathbb{R}$. Suppose that (a_n) is bounded and monotone. Then (a_n) converges.

Proof

Assume (a_n) is bounded and monotone. Since (a_n) is bounded, the set

$$A := \{a_n \,:\, n \in \mathbb{N}\} \subseteq \mathbb{R}$$

is bounded below and above. By the Axiom of Completeness of $\mathbb{R}$ there exist $i, s \in \mathbb{R}$ such that

$$i = \inf A\,, \quad s = \sup A\,.$$

We have two cases:

1. (a_n) is increasing: We are going to prove that

$$\lim_{n\to\infty} a_n = s\,.$$

Equivalently, we need to prove that

$$\forall\, \varepsilon > 0\,,\, \exists\, N \in \mathbb{N} \text{ s.t. } \forall\, n \geq N\,,\; |a_n - s| < \varepsilon\,. \tag{5.5}$$

Let $\varepsilon > 0$. Since s is the smallest upper bound for A, this means

$$s - \varepsilon$$

is not an upper bound. Therefore there exists $N \in \mathbb{N}$ such that

$$s - \varepsilon < a_N\,. \tag{5.6}$$

Let $n \geq N$. Since a_n is increasing, we have

$$a_N \leq a_n\,, \quad \forall\, n \geq N\,. \tag{5.7}$$

Moreover s is the supremum of A, so that

$$a_n \leq s < s + \varepsilon\,, \quad \forall\, n \in \mathbb{N}\,. \tag{5.8}$$

Putting together estimates (5.6)-(5.7)-(5.8) we get

$$s - \varepsilon < a_N \leq a_n \leq s < s + \varepsilon\,, \quad \forall\, n \geq N\,.$$

The above implies

$$s - \varepsilon < a_n < s + \varepsilon\,, \quad \forall\, n \geq N\,,$$

which is equivalent to (5.5).

2. (a_n) is decreasing: With a similar proof, one can show that

$$\lim_{n\to\infty} a_n = i\,.$$

This is left as an exercise.

5.3 Example: Euler's Number

As an application of the Monotone Convergence Theorem we can give a formal definition for the Euler's Number

$$e = 2.71828182845904523536\ldots$$

Theorem 5.36

Consider the sequence

$$a_n = \left(1 + \frac{1}{n}\right)^n\,.$$

We have that:

1. (a_n) is monotone increasing,
2. (a_n) is bounded.

In particular (a_n) is convergent.

Proof

Part 1. We prove that (a_n) is increasing

$$a_n \geq a_{n-1}\,, \quad \forall\, n \in \mathbb{N}\,,$$

which by definition is equivalent to

$$\left(1 + \frac{1}{n}\right)^n \geq \left(1 + \frac{1}{n-1}\right)^{n-1}\,, \quad \forall\, n \in \mathbb{N}\,.$$

Summing the fractions we get

$$\left(\frac{n+1}{n}\right)^n \geq \left(\frac{n}{n-1}\right)^{n-1}\,.$$

Multiplying by $((n-1)/n)^n$ we obtain

$$\left(\frac{n-1}{n}\right)^n \left(\frac{n+1}{n}\right)^n \geq \frac{n-1}{n}\,,$$

which simplifies to

$$\left(1 - \frac{1}{n^2}\right)^n \geq 1 - \frac{1}{n}\,, \quad \forall\, n \in \mathbb{N}\,. \tag{5.9}$$

Therefore (a_n) is increasing if and only if (5.9) holds. Recall Bernoulli's inequality from Lemma ??: For $x \in \mathbb{R}$, $x > -1$, it holds

$$(1 + x)^n \geq 1 + nx\,, \quad \forall\, n \in \mathbb{N}\,.$$

Appliying Bernoulli's inequality with

$$x = -\frac{1}{n^2}$$

yields

$$\left(1 - \frac{1}{n^2}\right)^n \geq 1 + n\left(-\frac{1}{n^2}\right) = 1 - \frac{1}{n}\,,$$

which is exactly (5.9). Then (a_n) is increasing.
Part 2. We have to prove that (a_n) is bounded, that is, that there exists $M > 0$ such that

$$|a_n| \leq M\,, \quad \forall\, n \in \mathbb{N}\,.$$

To this end, introduce the sequence (b_n) by setting

$$b_n := \left(1 + \frac{1}{n}\right)^{n+1}\,.$$

The sequence (b_n) is decreasing.

To prove (b_n) is decreasing, we need to show that

$$b_{n-1} \geq b_n\,, \quad \forall\, n \in \mathbb{N}\,.$$

By definition of b_n, the above reads

$$\left(1 + \frac{1}{n-1}\right)^{n} \geq \left(1 + \frac{1}{n}\right)^{n+1}\,, \quad \forall\, n \in \mathbb{N}\,.$$

Summing the terms inside the brackets, the above is equivalent to

$$\left(\frac{n}{n-1}\right)^{n} \geq \left(\frac{n+1}{n}\right)^{n}\left(\frac{n+1}{n}\right)\,.$$

Multiplying by $(n/(n+1))^n$ we get

$$\left(\frac{n^2}{n^2-1}\right)^{n} \geq \left(\frac{n+1}{n}\right)\,.$$

The above is equivalent to

$$\left(1 + \frac{1}{n^2-1}\right)^{n} \geq \left(1 + \frac{1}{n}\right)\,. \tag{5.10}$$

Therefore (b_n) is decreasing if and only if (5.10) holds for all $n \in \mathbb{N}$. By choosing

$$x = \frac{1}{n^2-1}$$

in Bernoulli's inequality, we obtain

$$\begin{aligned}\left(1 + \frac{1}{n^2-1}\right)^{n} &\geq 1 + n\left(\frac{1}{n^2-1}\right) \\ &= 1 + \frac{n}{n^2-1} \\ &\geq 1 + \frac{1}{n}\,,\end{aligned}$$

where in the last inequality we used that

$$\frac{n}{n^2-1} > \frac{1}{n}\,,$$

which holds, being equivalent to $n^2 > n^2 - 1$. We have therefore proven (5.10), and hence (b_n) is decreasing.

We now observe that For all $n \in \mathbb{N}$

$$\begin{aligned} b_n &= \left(1 + \frac{1}{n}\right)^{n+1} \\ &= \left(1 + \frac{1}{n}\right)^{n}\left(1 + \frac{1}{n}\right) \\ &= a_n\left(1 + \frac{1}{n}\right) \\ &> a_n\,. \end{aligned}$$

Since (a_n) is increasing and (b_n) is decreasing, in particular

$$a_n \geq a_1\,, \quad b_n \leq b_1\,.$$

Therefore

$$a_1 \leq a_n < b_n \leq b_1\,, \quad \forall\, n \in \mathbb{N}\,.$$

We compute

$$a_1 = 2\,, \quad b_1 = 4\,,$$

from which we get

$$2 \leq a_n \leq 4\,, \quad \forall\, n \in \mathbb{N}\,.$$

Therefore

$$|a_n| \leq 4\,, \quad \forall\, n \in \mathbb{N}\,,$$

showing that (a_n) is bounded.
Part 3. The sequence (a_n) is increasing and bounded above. Therefore (a_n) is convergent by the Monotone Convergence Theorem 5.35.

Thanks to Theorem 5.36 we can define the Euler's Number e.

Definition 5.37: Euler's Number

The Euler's number is defined as

$$e := \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^{n}\,.$$

Setting $n = 1000$ in the formula for (a_n), we get an approximation of e:

$$e \approx a_{1000} = 2.7169\,.$$

5.4 Some important limits

In this section we investigate limits of some sequences to which the Limit Tests do not apply.

Theorem 5.38

Let $x \in \mathbb{R}$, with $x > 0$. Then

$$\lim_{n\to\infty} \sqrt[n]{x} = 1\,.$$

Proof

Step 1. Assume $x \geq 1$. In this case

$$\sqrt[n]{x} \geq 1\,.$$

Define

$$b_n := \sqrt[n]{x} - 1\,,$$

so that $b_n \geq 0$. By Bernoulli's Inequality we have

$$x = (1+b_n)^n \geq 1 + nb_n\,.$$

Therefore

$$0 \leq b_n \leq \frac{x-1}{n}\,.$$

Since

$$\frac{x-1}{n} \longrightarrow 0\,,$$

by the Squeeze Theorem we infer $b_n \to 0$, and hence

$$\sqrt[n]{x} = 1 + b_n \longrightarrow 1 + 0 = 1\,,$$

by the Algebra of Limits.

Step 2. Assume $0 < x < 1$. In this case

$$\frac{1}{x} > 1\,.$$

Therefore

$$\lim_{n\to\infty} \sqrt[n]{1/x} = 1\,.$$

by Step 1. Therefore

$$\sqrt[n]{x} = \frac{1}{\sqrt[n]{1/x}} \longrightarrow \frac{1}{1} = 1\,,$$

by the Algebra of Limits.

Theorem 5.39

Let (a_n) be a sequence such that $a_n \to 0$. Then

$$\sin(a_n) \to 0\,, \quad \cos(a_n) \to 1\,.$$

Proof

Assume that $a_n \to 0$ and set

$$\varepsilon := \frac{\pi}{2}\,.$$

By the convergence $a_m \to 0$ there exists $N \in \mathbb{N}$ such that

$$|a_n| < \varepsilon = \frac{\pi}{2} \quad \forall\, n \geq N\,. \tag{5.11}$$

Step 1. We prove that

$$\sin(a_n) \to 0\,.$$

By elementary trigonometry we have

$$0 \leq |\sin(x)| = \sin|x| \leq |x|\,, \quad \forall\, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\,.$$

Therefore, since (5.11) holds, we can substitute $x = a_n$ in the above inequality to get

$$0 \leq |\sin(a_n)| \leq |a_n|\,, \quad \forall\, n \geq \mathbb{N}\,.$$

Since $a_n \to 0$, we also have $|a_n| \to 0$. Therefore $|\sin(a_n)| \to 0$ by the Squeeze Theorem. This immediately implies $\sin(a_n) \to 0$.

Step 2. We prove that

$$\cos(a_n) \to 1\,.$$

Inverting the relation

$$\cos^2(x) + \sin^2(x) = 1\,,$$

we obtain

$$\cos(x) = \pm\sqrt{1 - \sin^2(x)}\,.$$

We have that $\cos(x) \geq 0$ for $-\pi/2 \leq x \leq \pi/2$. Thus

$$\cos(x) = \sqrt{1-\sin^2(x)}\,, \quad \forall\, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\,.$$

Since (5.11) holds, we can set $x = a_n$ in the above inequality and obtain

$$\cos(a_n) = \sqrt{1 - \sin^2(a_n)}\,, \quad \forall\, n \geq N\,.$$

By Step 1 we know that $\sin(a_n) \to 0$. Therefore, by the Algebra of Limits,

$$1 - \sin^2(a_n) \longrightarrow 1 - 0 \cdot 0 = 1\,.$$

Using Theorem 5.20 we have

$$\cos(a_n) = \sqrt{1 - \sin^2(a_n)} \longrightarrow \sqrt{1} = 1\,,$$

concluding the proof.

Theorem 5.40

Suppose (a_n) is such that $a_n \to 0$ and

$$a_n \neq 0\,, \quad \forall\, n \in \mathbb{N}\,.$$

Then

$$\lim_{n\to\infty} \frac{\sin(a_n)}{a_n} = 1\,.$$

Proof

The following elementary trigonometric inequality holds:

$$\sin(x) < x < \tan(x)\,, \quad \forall\, x \in \left[0, \frac{\pi}{2}\right]\,.$$

Note that $\sin x > 0$ for $0 < x < \pi/2$. Therefore we can divide the above inequality by $\sin(x)$ and take the reciprocals to get

$$\cos(x) < \frac{\sin(x)}{x} < 1\,, \quad \forall\, x \in \left(0, \frac{\pi}{2}\right]\,.$$

If $-\pi/2 < x < 0$, we can apply the above inequality to $-x$ to obtain

$$\cos(-x) < \frac{\sin(-x)}{-x} < 1\,.$$

Recalling that $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$, we get

$$\cos(x) < \frac{\sin(x)}{x} < 1\,, \quad \forall\, x \in \left(-\frac{\pi}{2}, 0\right]\,.$$

Thus

$$\cos(x) < \frac{\sin(x)}{x} < 1\,, \quad \forall\, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}\,. \tag{5.12}$$

Let

$$\varepsilon := \frac{\pi}{2}\,.$$

Since $a_n \to 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n| < \varepsilon = \frac{\pi}{2}\,, \quad \forall\, n \geq N\,.$$

Since $a_n \neq 0$ by assumption, the above shows that

$$a_n \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}\,, \quad \forall\, n \geq \mathbb{N}\,.$$

Therefore we can substitute $x = a_n$ into (5.12) to get

$$\cos(a_n) < \frac{\sin(a_n)}{a_n} < 1\,, \quad \forall\, n \geq N\,.$$

We have

$$\cos(a_n) \to 1$$

by Theorem 5.39. By the Squeeze Theorem we conclude that

$$\lim_{n\to\infty} \frac{\sin(a_n)}{a_n} = 1\,.$$

Warning

You might be tempted to apply L'Hôpital's rule (which we did not cover in these Lecture Notes) to compute

$$\lim_{x\to 0} \frac{\sin(x)}{x}\,.$$

This would yield the correct limit

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{(\sin(x))'}{(x)'} = \lim_{x\to 0} \cos(x) = 1\,.$$

However this is a circular argument, since the derivative of $\sin(x)$ at $x = 0$ is defined as the limit

$$\lim_{x\to 0} \frac{\sin(x)}{x}\,.$$

Theorem 5.41

Suppose (a_n) is such that $a_n \to 0$ and

$$a_n \neq 0\,, \quad \forall\, n \in \mathbb{N}\,.$$

Then

$$\lim_{n\to\infty} \frac{1-\cos(a_n)}{(a_n)^2} = \frac{1}{2}\,, \quad \lim_{n\to\infty} \frac{1-\cos(a_n)}{a_n} = 0\,.$$

Proof

Step 1. By Theorem 5.39 and Theorem 5.40, we have

$$\cos(a_n) \to 1\,, \quad \frac{\sin(a_n)}{a_n} \to 1\,.$$

Therefore

$$\begin{aligned}
\frac{1-\cos(a_n)}{(a_n)^2} &= \frac{1-\cos(a_n)}{(a_n)^2} \, \frac{1+\cos(a_n)}{1+\cos(a_n)} \\
&= \frac{1-\cos^2(a_n)}{(a_n)^2} \, \frac{1}{1+\cos(a_n)} \\
&= \left(\frac{\sin(a_n)}{a_n}\right)^2 \frac{1}{1+\cos(a_n)} \longrightarrow 1 \cdot \frac{1}{1+1} = \frac{1}{2}\,,
\end{aligned}$$

where in the last line we use the Algebra of Limits.

Step 2. We have

$$\frac{1-\cos(a_n)}{a_n} = a_n \cdot \frac{1-\cos(a_n)}{(a_n)^2} \longrightarrow 0 \cdot \frac{1}{2} = 0\,,$$

using Step 1 and the Algebra of Limits.

Example 5.42

Question. Prove that

$$\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = 1\,. \tag{5.13}$$

Solution. This is because

$$n \sin\left(\frac{1}{n}\right) = \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} \longrightarrow 1\,,$$

by Theorem 5.40 with $a_n = 1/n$.

Example 5.43

Question. Prove that

$$\lim_{n\to\infty} n^2 \left(1 - \cos\left(\frac{1}{n}\right)\right) = \frac{1}{2}\,. \tag{5.14}$$

Solution. Indeed,

$$n^2 \left(1 - \cos\left(\frac{1}{n}\right)\right) = \frac{1 - \cos\left(\frac{1}{n}\right)}{\frac{1}{n^2}} \longrightarrow \frac{1}{2}\,,$$

by applying Theorem 5.41 with $a_n = 1/n$.

Example 5.44

Question. Prove that

$$\lim_{n\to\infty} \frac{n\left(1 - \cos\left(\frac{1}{n}\right)\right)}{\sin\left(\frac{1}{n}\right)} = \frac{1}{2}\,.$$

Solution. Using (5.14)-(5.13) and the Algebra of Limits

$$\begin{aligned} \frac{n\left(1 - \cos\left(\frac{1}{n}\right)\right)}{\sin\left(\frac{1}{n}\right)} &= \frac{n^2\left(1 - \cos\left(\frac{1}{n}\right)\right)}{n\sin\left(\frac{1}{n}\right)} \\ &\longrightarrow \frac{1/2}{1} = \frac{1}{2}\,. \end{aligned}$$

Example 5.45

Question. Prove that

$$\lim_{n\to\infty} n \cos\left(\frac{2}{n}\right) \sin\left(\frac{2}{n}\right) = 2\,.$$

Solution. We have

$$\cos\left(\frac{2}{n}\right) \longrightarrow 1\,,$$

by Theorem 5.39 applied with $a_n = 2/n$. Moreover

$$\frac{\sin\left(\frac{2}{n}\right)}{\frac{2}{n}} \longrightarrow 1\,,$$

by Theorem 5.40 applied with $a_n = 2/n$. Therefore

$$\begin{aligned} n \cos\left(\frac{2}{n}\right) \sin\left(\frac{2}{n}\right) &= 2 \cdot \cos\left(\frac{2}{n}\right) \cdot \frac{\sin\left(\frac{2}{n}\right)}{\frac{2}{n}} \\ &\longrightarrow 2 \cdot 1 \cdot 1 = 2\,, \end{aligned}$$

where we used the Algebra of Limits.

Example 5.46

Question. Prove that

$$\lim_{n\to\infty} \frac{n^2+1}{n+1} \sin\left(\frac{1}{n}\right) = 1\,.$$

Solution. Note that

$$\begin{aligned} \frac{n^2+1}{n+1} \sin\left(\frac{1}{n}\right) &= \left(\frac{1 + \frac{1}{n^2}}{1 + \frac{1}{n}}\right) \cdot \left(n \sin\left(\frac{1}{n}\right)\right) \\ &\longrightarrow \frac{1+0}{1+0} \cdot 1 = 1\,, \end{aligned}$$

where we used (5.13) and the Algebra of Limits.

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