Numbers, Sequences and Series

Lecture Notes

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Welcome

These are the Lecture Notes of **Numbers, Sequences and Series 400297** for T1 2023/24 at the University of Hull. I will follow these lecture notes during the course. If you have any question or find any typo, please email me at

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Up to date information about the course, Tutorials and Homework will be published on the University of Hull **Canvas Website**

canvas.hull.ac.uk/courses/67551

and on the **Course Webpage** hosted on my website

silviofanzon.com/blog/2023/NSS

Digital Notes

Digital version of these notes available at

silviofanzon.com/2023-NSS-Notes

Readings

We will study the set of real numbers $\mathbb{R}$, and then sequences and series in $\mathbb{R}$. I will follow mainly the textbook by Bartle and Sherbert [2]. Another good reading is the book by Abbott [1]. I also point out the classic book by Rudin [3], although this is more difficult to understand.

> **!** You are not expected to purchase any of the above books. These lecture notes will cover 100% of the topics you are expected to known in order to excel in the final exam.

1 Introduction

The first aim of this lecture notes is to rigorously introduce the set of **real numbers**, which is denoted by $\mathbb{R}$. But what do we mean by real numbers? To start our discussion, introduce the set of natural numbers (or non-negative integers)

$$\mathbb{N} = \{0, 1, 2, 3, 4, 5, \ldots\}$$

On this set we have a notion of **sum** of two numbers, denoted as usual by

$$n + m$$

for $n, m \in \mathbb{N}$. Here the symbol $\in$ denotes that m and n belong to $\mathbb{N}$. For example $3 + 7$ results in 10.

Question 1.1

Can the sum be inverted? That is, given any $n, m \in \mathbb{N}$, can you always find $x \in \mathbb{N}$ such that

$$n + x = m\,? \tag{1.1}$$

Of course to invert (1.1) we can just perform a **subtraction**, implying that

$$x = m - n\,.$$

But there is a catch. In general x does not need to be in $\mathbb{N}$. For example, take $n = 10$ and $m = 1$. Then $x = -9$, which does not belong to $\mathbb{N}$. Therefore the answer to Question 1.1 is **NO**.

To make sure that we can always invert the sum, we need to **extend** the set $\mathbb{N}$. This is done simply by introducing the set of **integers**

$$\mathbb{Z} := \{-n, n \ : \ n \in \mathbb{N}\}\,,$$

that is, the set

$$\mathbb{Z} := \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}\,.$$

The sum can be extended to $\mathbb{Z}$, by defining

$$(-n) + (-m) := -(m + n) \tag{1.2}$$

for all $m, n \in \mathbb{N}$. Now every element of $\mathbb{Z}$ possesses an **inverse**, that is, for each $n \in \mathbb{Z}$, there exists $m \in \mathbb{Z}$, such that

$$n + m = 0 \,.$$

Can we characterize m explicitly? Of course! Seeing the definition at (1.2), we simply have

$$m = -n \,.$$

On the set $\mathbb{Z}$ we can also define the operation of **multiplication**, in the usual way we learnt in school. For $n, m \in \mathbb{Z}$, we denote the multiplication by nm or $n \cdot m$. For example $7 \cdot 2 = 14$ and $1 \cdot (-1) = -1$.

Question 1.2

Can the multiplication in $\mathbb{Z}$ be inverted? That is, given any $n, m \in \mathbb{Z}$, can you always find $x \in \mathbb{Z}$ such that

$$nx = m \,? \tag{1.3}$$

To invert (1.3) if $n \neq 0$, we can just perform a **division**, to obtain

$$x = \frac{m}{n} \,.$$

But again there is a catch. Indeed taking $n = 2$ and $m = 1$ yields $x = 1/2$, which does not belong to $\mathbb{Z}$. The answer to Question 1.2 is therefore **NO**.

Thus, in order to invert the multiplication, we need to **extend** the set of integers $\mathbb{Z}$. This extension is called the set of **rational numbers**, defined by

$$\mathbb{Q} := \left\{ \frac{m}{n} \; : \; m, n \in \mathbb{Z}, \, n \neq 0 \right\} \,.$$

We then extend the operations of sum and multiplication to $\mathbb{Q}$ by defining

$$\frac{m}{n} + \frac{p}{q} := \frac{mq + np}{nq}$$

and

$$\frac{m}{n} \cdot \frac{p}{q} := \frac{mp}{nq}$$

Now the multiplication is invertible in $\mathbb{Q}$. Specifically, each non-zero element has an inverse: the inverse of m/n is given by n/m.

To summarize, we have extended $\mathbb{N}$ to $\mathbb{Z}$, and $\mathbb{Z}$ to $\mathbb{Q}$. By construction we have

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \,.$$

Moreover **sum** and **product** are **invertible** in $\mathbb{Q}$. Now we are happy right? So and so.

Question 1.3

Can we draw the set $\mathbb{Q}$?

It is clear how to draw $\mathbb{Z}$, as seen below.
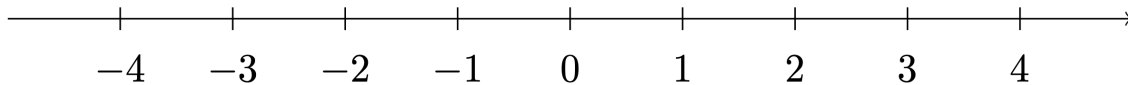
Figure 1.1: Representation of integers $\mathbb{Z}$

However $\mathbb{Q}$ is much **larger** than the set $\mathbb{Z}$ represented by the ticks in Figure 1.1. What do we mean by **larger**? For example, consider $0 \in \mathbb{Q}$.

Question 1.4

What is the number $x \in \mathbb{Q}$ which is closest to 0?

There is no right answer to the above question, since whichever rational number m/n you consider, you can always squeeze the rational number $m/(2n)$ in between:

$$0 < \frac{m}{2n} < \frac{m}{n} \,.$$

For example think about the case of the numbers

$$\frac{1}{n} \ \text{ for } \ n \in \mathbb{N}, \, n \neq 0 \,.$$

Such numbers get arbitrarily close to 0, as depicted below.
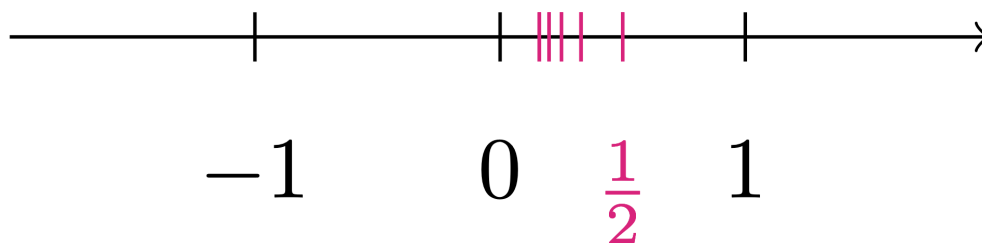
Figure 1.2: Fractions $\frac{1}{n}$ can get arbitrarily close to 0

Maybe if we do the same reasoning with other progressively smaller rational numbers, we manage to fill out the interval $[0, 1]$. In other words, we might conjecture the following.

Conjecture 1.5

Maybe $\mathbb{Q}$ can be represented by a continuous line.

Do you think the above conjecture is true? If it was, mathematics would be quite boring. Indeed Conjecture 1.5 is **false**, as shown by the Theorem below.

Theorem 1.6

The number $\sqrt{2}$ does not belong to $\mathbb{Q}$.

Theorem 1.6 is the reason why $\sqrt{2}$ is called an **irrational number**. For reference, a few digits of $\sqrt{2}$ are given by

$$\sqrt{2} = 1.414213562373095048\ldots$$

and the situation is as in the picture below.
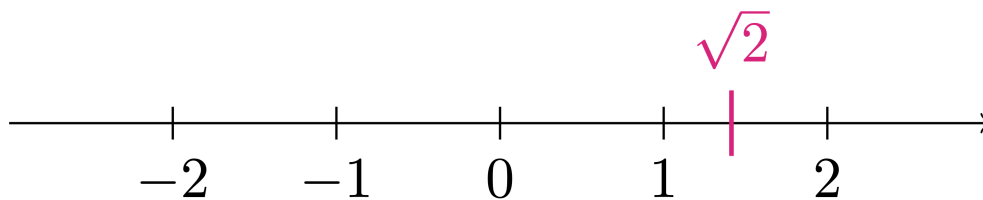
Figure 1.3: Representing $\sqrt{2}$ on the numbers line.

We can therefore see that Conjecture 1.5 is **false**, and $\mathbb{Q}$ is not a line: indeed $\mathbb{Q}$ has a **gap** at $\sqrt{2}$. Let us see why Theorem 1.6 is true.

Proof: Proof of Theorem 1.6

We prove that

$$\sqrt{2} \notin \mathbb{Q}$$

by **contradiction**.
Wait, what does this mean? Proving the claim by contradiction means assuming that the claim is **false**. This means we **assume** that

$$\sqrt{2} \in \mathbb{Q}. \tag{1.4}$$

From this assumption we then start deducing other statements, hoping to encounter a statement which is **FALSE**. But if (1.4) leads to a false statement, then it must be that (1.4) is **FALSE**. Thus the contrary of (1.4) must hold, meaning that

$$\sqrt{2} \notin \mathbb{Q}$$

as we wanted to show. This would conclude the proof.
Now we need to actually show that (1.4) will lead to a contradiction. Since this is our first proof, let us take it slowly, step-by-step.

1. Assuming (1.4) just means that there exists $q \in \mathbb{Q}$ such that

$$q = \sqrt{2} \,. \tag{1.5}$$

2. Since $q \in \mathbb{Q}$, by definition we have

$$q = \frac{m}{n}$$

   for some $m, n \in \mathbb{N}$ with $n \neq 0$.

3. Recalling (1.5), we then have

$$\frac{m}{n} = \sqrt{2} \,.$$

4. We can square the above equation to get

$$\frac{m^2}{n^2} = 2 \,. \tag{1.6}$$

5. **Withouth loss of generality**, we can **assume** that m and n have no common factors.

   Wait. What does Step 5 mean? You will encounter the sentence *withouth loss of generality* many times in mathematics. It is often abbreviated in `WLOG`. WLOG means that the assumption that follows is chosen arbitrarily, but does not affect the validity of the proof in general.
   For example in our case we can assume that m and n have no common factor. This is because if m and n had common factors, then it would mean

$$m = a\tilde{m} \,, \quad n = a\tilde{n}$$

   for some $a \in \mathbb{N}$ with $a \neq 0$. Then

$$\frac{m}{n} = \frac{a\tilde{m}}{a\tilde{n}} = \frac{\tilde{m}}{\tilde{n}} \,.$$

   Therefore by (1.6)

$$\frac{\tilde{m}^2}{\tilde{n}^2} = 2 \,.$$

   The proof can now proceed in the same way we would have proceeded from Step 4, but in addition we have the hypothesis that $\tilde{m}$ and $\tilde{n}$ have no common factors.

6. Equation (1.6) implies

$$m^2 = 2n^2 \,. \tag{1.7}$$

   Therefore the integer m^2 is an even number.

Why is m^2 even? As you already know, **even** numbers are

$$0, 2, 4, 6, 8, 10, 12, \ldots$$

All these numbers have in common that they can be divided by 2, and so they can be written as

$$2p$$

for some $p \in \mathbb{N}$. For example 52 is even, because

$$52 = 2 \cdot 26 \,.$$

Instead, **odd** numbers are

$$1, 3, 5, 7, 8, 9, 11, \ldots$$

These can be all written as

$$2p + 1$$

for some $p \in \mathbb{N}$. For example 53 is odd, because

$$52 = 2 \cdot 26 + 1 \,.$$

7. Thus m is an even number, and so there exists $p \in \mathbb{N}$ such that

$$m = 2p \,. \tag{1.8}$$

Why is (1.8) true? Let us see what happens if we take the square of an even number $m = 2p$

$$m^2 = (2p)^2 = 4p^2 = 2(2p^2) = 2q \,.$$

Thus $m^2 = 2q$ for some $q \in \mathbb{N}$, and so m^2 is an even number. If instead m is odd, then $m = 2p + 1$ and

$$m^2 = (2p + 1)^2 = 4p^2 + 4p + 1 = 2(2p^2 + 2p) + 1$$

showing that also m^2 is odd.
This justifies Step 7: Indeed we know that m^2 is an even number from Step 6. If m was odd, then m^2 would be odd. Hence m must be even as well.

8. If we substitute (1.8) in (1.7) we get

$$m^2 = 2n^2 \implies (2p)^2 = 2n^2 \implies 4p^2 = 2n^2$$

Dividing both terms by 2, we obtain

$$n^2 = 2p^2 \,. \tag{1.9}$$

9. We now make a series of observations:

- Equation (1.9) says that n^2 is even.
- Step 6 says that m^2 is even.
- Therefore n and m are also even.

- Hence n and m have 2 as common factor.
- But Step 5 says that n and m have no common factors.
- `CONTRADICTION`

10. Our reasoning has run into a **contradiction**, starting from assumption (1.4), which says that

$$\sqrt{2} \in \mathbb{Q}\,.$$

Hence the above must be **FALSE**, and so

$$\sqrt{2} \notin \mathbb{Q}$$

ending the proof.

Seeing that $\sqrt{2} \notin \mathbb{Q}$, we might be tempted to just fill in the gap by adding $\sqrt{2}$ to $\mathbb{Q}$. However, with analogous proof to Theorem 1.6, we can prove that

$$\sqrt{p} \notin \mathbb{Q}$$

for each prime number p. As there are infinite prime numbers, this means that $\mathbb{Q}$ has infinite gaps. Then we might attempt to fill in these gaps via the extension

$$\tilde{\mathbb{Q}} := \mathbb{Q} \cup \{\sqrt{p} : \ p \ \text{prime}\}\,.$$

However even this is not enough, as we would still have numbers which are not contained in $\tilde{\mathbb{Q}}$, for example

$$\sqrt{2} + \sqrt{3},\ \pi,\ \pi + \sqrt{2} \notin \tilde{\mathbb{Q}}\,.$$

Remark 1.7

Proving that

$$\sqrt{2} + \sqrt{3} \notin \mathbb{Q}$$

is relatively easy, and will be left as an **exercise**. Instead, proving that

$$\pi \notin \mathbb{Q}$$

is way more complicated. There are several proof of the fact, all requiring mathematics which is more advanced of the one presented in this course. For some proofs, see this Wikipedia page.

The reality of things is that to **complete** $\mathbb{Q}$ and make it into a **continuous line** we have to add a lot of points. Indeed, we need to add way more points than the ones already contained in $\mathbb{Q}$. Such extension of $\mathbb{Q}$ will be called $\mathbb{R}$, the set of **real numbers**. The inclusions will therefore be

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

The set $\mathbb{R}$ is not at all trivial to construct. In fact, at first we will not construct it, but just do the following:

- We will assume that $\mathbb{R}$ **exists** and satisfies some basic **axioms**.
- One of the axioms is that $\mathbb{R}$ fills **all** the **gaps** that $\mathbb{Q}$ has. Therefore $\mathbb{R}$ can be thought as a **continuous** line.
- We will study the **properties** of $\mathbb{R}$ which descend from such **axioms**.

For example one of the properties of $\mathbb{R}$ will be the following:

Theorem 1.8:   We will prove this in the future

$\mathbb{R}$ contains all the square roots. This means that for every $x \in \mathbb{R}$ with $x \geq 0$, we have

$$\sqrt{x} \in \mathbb{R}.$$

At the end of this chapter we will provide a concrete **model** for the real numbers $\mathbb{R}$, to prove once and for all that such set indeed exists.

Theorem 1.9:   We will prove this in the future

There exists a set $\mathbb{R}$, called the set of real numbers, which has the following properties:

- $\mathbb{R}$ extends $\mathbb{Q}$, that is,
$$\mathbb{Q} \subset \mathbb{R}.$$

- $\mathbb{R}$ satisfies certain **axioms**.
- $\mathbb{R}$ fills **all** the **gaps** that $\mathbb{Q}$ has. In particular $\mathbb{R}$ can be represented by a **continuous** line.

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2 Preliminaries

Before introducing $\mathbb{R}$ we want to make sure that we cover all the basics needed for the task.

2.1 Sets

A sets is a **collection** of objects. These objects are called **elements** of the set. For example in the previous section we mentioned the following sets:

- $\mathbb{N}$ the set of natural numbers
- $\mathbb{Z}$ the set of integers
- $\mathbb{Q}$ the set of rational numbers
- $\mathbb{R}$ the set of real numbers

Given an arbitrary set A, we write

$$x \in A$$

if the element x belongs to the set A. If an element x is not contained in A, we say that

$$x \notin A.$$

Remark 2.1

A set can contain all sorts of elements. For example the students in a classroom can be modelled by a set S. The elements of the set are the students. For example

$$S = \{\text{Alice, Olivia, Jake, Sahab}\}$$

In this case we have

$$\text{Alice} \in S$$

but instead

$$\text{Silvio} \notin S.$$

2.2 Logic

In this section we introduce some basic logic symbols. Suppose that you are given two statements, say α and β. The formula

$$\alpha \implies \beta$$

means that α **implies** β. In other words, if α is true then also β is true.
The formula

$$\alpha \impliedby \beta$$

means that α is implied by β: if β is true then also α is true.
When we write

$$\alpha \iff \beta \tag{2.1}$$

we mean that α and β are **equivalent**. Note that (2.1) is equivalent to

$$\alpha \implies \beta \ \text{ and } \ \beta \implies \alpha \,.$$

Such equivalence is very useful in proofs.

Example 2.2

We have that

$$x > 0 \implies x > -100 \,,$$

and

$$\text{contradiction} \impliedby \sqrt{2} \in \mathbb{Q} \,.$$

Concerning $\iff$ we have

$$x^2 < 2 \iff -\sqrt{2} < x < \sqrt{2} \,.$$

We now introduce logic **quantifiers**. These are

- $\forall$ which reads **for all**
- $\exists$ which reads **exists**
- $\exists!$ which reads **exists unique**
- $\nexists$ which reads **does not exists**

These work in the following way. Suppose that you are given a statement $\alpha(x)$ which depends on the point $x \in \mathbb{R}$. Then we say

- $\alpha(x)$ is satisfied for all $x \in A$ with A some collection of numbers. This translates to the symbols

$$\alpha(x) \ \text{ is true } \ \forall\, x \in A \,,$$

- There exists some x in $\mathbb{R}$ such that $\alpha(x)$ is satisfied: in symbols

$$\exists\, x \in \mathbb{R} \ \text{ such that } \ \alpha(x) \ \text{ is true,}$$

- There exists a unique x_0 in $\mathbb{R}$ such that $\alpha(x)$ is satisfied: in symbols

$$\exists! \, x_0 \in \mathbb{R} \ \text{ such that } \ \alpha(x_0) \ \text{ is true,}$$

- $\alpha(x)$ is never satisfied:

$$\nexists \, x \in \mathbb{R} \ \text{ such that } \ \alpha(x) \ \text{ is true.}$$

Example 2.3

Let us make concrete examples:

- The expression x^2 is always non-negative. Thus we can say

$$x^2 \geq 0 \ \text{ for all } \ x \in \mathbb{R} \,.$$

- The equation $x^2 = 1$ has two solutions $x = 1$ and $x = -1$. Therefore we can say

$$\exists \, x \in \mathbb{R} \text{ such that } \ x^2 = 1 \,.$$

- The equation $x^3 = 1$ has a unique solution $x = 1$. Thus

$$\exists! \, x \in \mathbb{R} \text{ such that } \ x^3 = 1 \,.$$

- We know that the equation $x^2 = 2$ has no solutions in $\mathbb{Q}$. Then

$$\nexists \, x \in \mathbb{Q} \text{ such that } \ x^2 = 2 \,.$$

2.3 Operations on sets

2.3.1 Union and intersection

For two sets A and B we define their **union** as the set

$$A \cup B := \{x : \ x \in A \ \text{ or } \ x \in B\} \,.$$

The **intersection** of A and B is defined by

$$A \cap B := \{x : \ x \in A \ \text{ and } \ x \in B\} \,.$$

We denote the **empty set** by the symbol $\varnothing$. Two sets are **disjoint** if

$$A \cap B = \varnothing \,.$$

Example 2.4

Define the subset of rational numbers

$$S := \left\{ x \in \mathbb{Q} \; : \; 0 < x < \frac{5}{2} \right\}.$$

Then we have

$$\mathbb{N} \cap S = \{1, 2\}.$$

We can also define the sets of **even** and **odd** numbers by

$$E := \{2n \; : \; n \in \mathbb{N}\}, \tag{2.2}$$
$$O := \{2n + 1 \; : \; n \in \mathbb{N}\}. \tag{2.3}$$

Then we have

$$\mathbb{N} \cap E = E, \quad \mathbb{N} \cap O = O, \tag{2.4}$$
$$O \cup E = \mathbb{N}, \quad O \cap D = \emptyset. \tag{2.5}$$

2.3.2 Inclusion and equality

Given two sets A and B, we say that A is **contained** in B if all the elements of A are also contained in B. This will be denoted with the **inclusion** symbol $\subseteq$, that is,

$$A \subseteq B.$$

In this case we say that

- A is a **subset** of B,
- B is a **superset** of A.

The inclusion $A \subseteq B$ is equivalent to the implication:

$$x \in A \implies x \in B$$

for all $x \in A$. The symbol $\implies$ reads **implies**, and denotes the fact that the first condition implies the second.

Example 2.5

Given two sets A and B we always have

$$(A \cap B) \subseteq A, \ (A \cap B) \subseteq B, \tag{2.6}$$
$$A \subseteq (A \cup B), \ B \subseteq (A \cup B). \tag{2.7}$$

We say that two sets A and B are equal if they contain the **same** elements. We denote equality by the symbol

$$A = B.$$

If $A \subseteq B$ and $A \neq B$, we write

$$A \subset B \quad \text{or} \quad A \subsetneq B.$$

Example 2.6

1. The sets
$$A = \{1, 2, 3\}, \quad B = \{3, 1, 2\}$$
are equal, that is $A = B$. This is because they contain exactly the same elements: **order** does not matter when talking about sets.

2. Consider the sets
$$A = \{1, 2\}, \quad B = \{1, 2, 5\}.$$
Then A is contained in B, but A is not equal to B. Therefore we write $A \subset B$ or $A \subsetneq B$.

Proposition 2.7

Let A and B be sets. Then
$$A = B$$
if and only if
$$A \subseteq B \ \text{and} \ B \subseteq A.$$

Proof

The proof is almost trivial. However it is a good exercise in basic logic, so let us do it.

1. First implication $\implies$:
Suppose that $A = B$. Let us show that $A \subseteq B$. Since $A = B$, this means that all the elements of A are also contained in B. Therefore if we take $x \in A$ we have

$$x \in A \implies x \in B.$$

This shows $A \subseteq B$. The proof of $B \subseteq A$ is similar.

2. Second implication $\Longleftarrow$:
   Suppose that $A \subseteq B$ and $B \subseteq A$. We need to show $A = B$, that is, A and B have the same elements. To this end let $x \in A$. Since $A \subseteq B$ then we have $x \in B$. Thus B contains all the elements of A. Since we are also assuming $B \subseteq A$, this means that A contains all the elements of B. Hence A and B contain the same elements, and $A = B$.

The above proposition is very useful when we need to **prove** that two sets are equal: rather than showing directly that $A = B$, we can prove that $A \subseteq B$ and $B \subseteq A$.

2.3.3 Infinite unions and intersections

Suppose given a set Ω, and a family of sets $A_n \subseteq \Omega$, where $n \in \mathbb{N}$. Then we can define the **infinte union**

$$\bigcup_{n \in \mathbb{N}} A_n := \{x \in \Omega : \ x \in A_n \ \text{for at least one} \ n \in \mathbb{N}\}.$$

The **infinte intersection** is defined as

$$\bigcap_{n \in \mathbb{N}} A_n := \{x \in \Omega : \ x \in A_n \ \text{for all} \ n \in \mathbb{N}\}.$$

Example 2.8

Let the ambient set be $\Omega := \mathbb{N}$ and define the family A_n by

$$A_1 := \{1, 2, 3, 4, ...\} \tag{2.8}$$
$$A_2 := \{2, 3, 4, 5, ...\} \tag{2.9}$$
$$A_3 := \{3, 4, 5, 6, ...\} \tag{2.10}$$
$$...... \tag{2.11}$$
$$A_n := \{n, n + 1, n + 2, n + 3, ...\}, \tag{2.12}$$

for arbitrary $n \in \mathbb{N}$. Then

$$\bigcup_{n \in \mathbb{N}} A_n = \mathbb{N}. \tag{2.13}$$

The above equality can be easily proven using Proposition 2.7. Indeed, assume that $m \in \cup_n A_n$. Then $m \in A_n$ for at least one $n \in \mathbb{N}$. Since $A_n \subseteq \mathbb{N}$, we conclude that $m \in \mathbb{N}$. This shows

$$\bigcup_{n \in \mathbb{N}} A_n \subseteq \mathbb{N}.$$

Conversely, suppose that $m \in \mathbb{N}$. By definition $m \in A_m$. Hence there exists at least one index n, $n = m$ in this case, such that $m \in A_n$. Then by definition $m \in \cup_{n \in \mathbb{N}} A_n$, showing that

$$\mathbb{N} \subseteq \bigcup_{n \in \mathbb{N}} A_n.$$

Hence we conclude (2.13) by Proposition 2.7.
We also have that
$$\bigcap_{n\in\mathbb{N}} A_n = \emptyset.$$ (2.14)

We prove the above by **contradiction**. Indeed, suppose that (2.14) is false, i.e.,
$$\bigcap_{n\in\mathbb{N}} A_n \neq \emptyset.$$

This means there exists some $m \in \mathbb{N}$ such that $m \in \cap_{n\in\mathbb{N}} A_n$. Hence, by definition, $m \in A_n$ for all $n \in \mathbb{N}$. However $m \notin A_{m+1}$, yielding a contradiction. Thus (2.14) holds.

2.3.4 Complement

Suppose that A and B are subsets of a larger set Ω. The **complement** of A with respect to B is the set of elements of B which do not belong to A, that is
$$B \setminus A := \{x \in \Omega : \ x \in B \text{ and } x \notin A\}.$$

In particular, the complement of A with respect to Ω is denoted by
$$A^c := \Omega \setminus A := \{x \in \Omega : \ x \notin A\}.$$

Remark 2.9

Suppose that $A \subseteq \Omega$. Then A and A^c form a **partition** of Ω, in the sense that
$$A \cup A^c = \Omega \ \text{ and } \ A \cap A^c = \emptyset.$$

Example 2.10

Suppose $A, B \subseteq \Omega$. Then
$$A \subseteq B \iff B^c \subseteq A^c.$$

Let us prove the above claim:

- First implication $\implies$:
  Suppose that $A \subseteq B$. We need to show that $B^c \subseteq A^c$. Hence, assume $x \in B^c$. By definition this means that $x \notin B$. Now notice that we cannot have that $x \in A$. Indeed, assume $x \in A$. By assumption we have $A \subseteq B$, hence $x \in B$. But we had assumed $x \in B$, contradiction. Therefore it must be that $x \notin A$. Thus $B^c \subseteq A^c$.

- Second implication $\Longleftarrow$:
  Essentially the same proof, hence we omit it.

We conclude by stating the De Morgan's Laws. The proof will be left as an exercise.

Proposition 2.11: De Morgan's Laws

Suppose $A, B \subseteq \Omega$. Then

$$(A \cap B)^c = A^c \cup B^c$$

and

$$(A \cup B)^c = A^c \cap B^c .$$

2.3.5 Power set

Let Ω be an arbitray set. We define the **power set** of Ω as

$$\mathscr{P}(\Omega) := \{A : A \subseteq \Omega\},$$

that is, the power set of Ω is the set of all subsets of Ω.

Remark 2.12

It holds that:

1. $\mathscr{P}(\Omega)$ is always non-empty, since we have that

$$\emptyset \in \mathscr{P}(\Omega), \quad \Omega \in \mathscr{P}(\Omega).$$

2. Given $A, B \in \mathscr{P}(\Omega)$, then the sets

$$A \cup B, \quad A \cap B, \quad A^c, \quad B \setminus A$$

are all elements of $\mathscr{P}(\Omega)$.

3. Suppose Ω is **discrete** and **finite**, that is,

$$\Omega = \{x_1, \ldots, x_m\}$$

for some $m \in \mathbb{N}$. Then $\mathscr{P}(\Omega)$ contains 2^m elements.

    This is because for each $x_i \in \Omega$ we have just two choices: either include x_i in a subset, or do not include x_i in a subset.

> **Example 2.13**
>
> Define the set
> $$\Omega = \{x, y, z\}.$$
>
> Then $\mathscr{P}(\Omega)$ has $2^3 = 8$ elements. These are
>
> - $\varnothing$
> - $\{x\}$
> - $\{y\}$
> - $\{z\}$
> - $\{x, y\}$
> - $\{x, z\}$
> - $\{y, z\}$
> - $\{x, y, z\}$
>
> We therefore write
>
> $$\mathscr{P}(\Omega) = \{\varnothing, \{x\}, \{y\}, \{z\}, \{x, y\} \tag{2.15}$$
> $$\{x, z\}, \{y, z\}, \{x, y, z\}\}. \tag{2.16}$$

2.3.6 Product of sets

Suppose A and B are two sets. The **product** of A and B is the set of pairs

$$A \times B := \{(a, b) \: : \: a \in A, \, b \in B\}.$$

By definition two elements in $A \times B$ are the same, in symbols

$$(a, b) = (\tilde{a}, \tilde{b})$$

if and only if they are equal component-by-componenent, that is

$$a = \tilde{a}, \qquad b = \tilde{b}.$$

2.4 Equivalence relation

Suppose A is a set. A **binary relation** R on A is a subset

$$R \subseteq A \times A.$$

Definition 2.14: Equivalence relation

A binary relation R is called an **equivalence relation** if it satisfies the following properties:

1. **Reflexive**: For each $x \in A$ one has
$$(x, x) \in R \,,$$
   This is saying that all the elements in A must be related to themselves

2. **Symmetric**: We have
$$(x, y) \in R \implies (y, x) \in R$$
   If x is related to y, then y is related to x

3. **Transitive**: We have
$$(x, y) \in R, \ (y, z) \in R \implies (x, z) \in R$$
   If x is related to y, and y is related to z, then x must be related to z

If $(x, y) \in R$ we write
$$x \sim y$$
and we say that x and y are **equivalent**.

Definition 2.15: Equivalence classes

Suppose R is an **equivalence relation** on A. The **equivalence class** of an element $x \in A$ is the set
$$[x] := \{y \in A : \ y \sim x\}\,.$$
The set of equivalence classes of elements of A with respect to the equivalence relation R is denoted by
$$A/R := \{[x] : \ x \in A\}\,.$$

Let us immediately clarify the above definitions by considering the prototypical example of equivalence relation: the **equality**.

Example 2.16: Equality is an equivalence relation

Consider the set of natural numbers $\mathbb{N}$. The equality defines a **binary relation** on $\mathbb{N} \times \mathbb{N}$, via
$$R := \{(x, y) \in \mathbb{N} \times \mathbb{N} : \ x = y\}\,.$$

Let us check that R is an **equivalence relation**:

1. Reflexive: It holds, since $x = x$ for all $x \in \mathbb{N}$,

2. Symmetric: Again $x = y$ if and only if $y = x$,

3. Transitive: If $x = y$ and $y = z$ then $x = z$.

The class of equivalence of $x \in \mathbb{N}$ is given by

$$[x] = \{x\},$$

that is, this relation is quite trivial, given that each element of $\mathbb{N}$ can only be related to itself. The quotient space is then

$$\mathbb{N}/R = \{[x] : x \in \mathbb{N}\} = \{\{x\} : x \in \mathbb{N}\}.$$

Example 2.17

Suppose that R is a binary relation on the set $\mathbb{Q}$ of rational numbers defined by

$$x \sim y \iff x - y \in \mathbb{Z}.$$

Then R is an equivalence relation on $\mathbb{Q}$. Indeed:

1. Reflexive: Let $x \in \mathbb{Q}$. Then $x - x = 0$ and $0 \in \mathbb{Z}$. Thus $x \sim x$.
2. Symmetric: If $x \sim y$ then $x - y \in \mathbb{Z}$. But then also

$$-(x - y) = y - x \in \mathbb{Z}$$

and so $y \sim x$.
3. Transitive: Suppose $x \sim y$ and $y \sim z$. Then

$$x - y \in \mathbb{Z} \text{ and } y - z \in \mathbb{Z}.$$

Thus we have

$$x - z = (x - y) + (y - z) \in \mathbb{Z}$$

showing that $x \sim z$. This shows that R is an equivalence relation on $\mathbb{Q}$.

Now note that

$$y \sim x \iff y - x \in \mathbb{Z}$$

and the above is equivalent to

$$\exists n \in \mathbb{Z} \text{ s.t. } y - x = n$$

which again is equivalent to

$$\exists n \in \mathbb{Z} \text{ s.t. } y = x + n.$$

Therefore all the elements of $\mathbb{Q}$ related to x by R are of the form

$$x + n, \forall n \in \mathbb{Z}.$$

The equivalence classes with respect to R are then

$$[x] = \{x + n \ : \ n \in \mathbb{Z}\}.$$

Each equivalence class has exactly one element in $[0, 1) \cap \mathbb{Q}$, meaning that:

$$\forall x \in \mathbb{Q}, \ \exists! \, q \in \mathbb{Q} \ \text{s.t.} \ 0 \leq q < 1 \ \text{and} \ q \in [x].$$

Therefore

$$\mathbb{Q}/R = \{[x] \ : \ x \in \mathbb{Q}\} = \{q \in \mathbb{Q} \ : \ 0 \leq q < 1\}.$$

2.5 Order relation

Similarly, we define **order relations**.

Definition 2.18: Partial order

A binary relation R on A is called a **partial order** if it satisfies the following properties:

1. **Reflexive**: For each $x \in A$ one has
$$(x, x) \in R,$$

2. **Transitive**: We have
$$(x, y) \in R, \ (y, z) \in R \implies (x, z) \in R$$

3. **Antisymmetric**: We have

$$(x, y) \in R \ \text{and} \ (y, x) \in R \implies x = y$$

   This is the only new condition with respect to the definition of equivalence relation, and it replaces symmetry.

Definition 2.19: Total order

A binary relation R on A is called a **total order relation** if it satisfies the following properties:

1. **Partial order**: R is a partial order on A.

2. **Total**: For each $x, y \in A$ we have

$$(x, y) \in R \ \text{or} \ (y, x) \in R.$$

   This is saying that all elements in A are related.

An example of partial order is the operation of **set inclusion**.

Example 2.20:   Set inclusion is a partial order

Consider an arbitrary non-empty set Ω and consider its **power set**

$$\mathscr{P}(\Omega) = \{A : A \subseteq \Omega\}.$$

The inclusion defines **binary relation** on $\mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$, via

$$R := \{(A, B) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) : A \subseteq B\}.$$

Let us check that R is an **order relation**:

1. Reflexive: It holds, since $A \subseteq A$ for all $A \in \mathscr{P}(\Omega)$,

2. Transitive: If $A \subseteq B$ and $B \subseteq C$, then by definition of inclusion $A \subseteq C$.

3. Antisymmetric: If $A \subseteq B$ and $B \subseteq A$, then $A = B$ by Proposition 2.7.

Therefore R is a **partial order** on $\mathscr{P}(\Omega)$. Note that in general R is **not** a total order. For example if we consider

$$\Omega = \{x, y\}.$$

Thus

$$\mathscr{P}(\Omega) = \{\varnothing, \{x\}, \{y\}, \{x, y\}\}.$$

If we pick $A = \{x\}$ and $B = \{y\}$ then $A \cap B = \varnothing$, meaning that

$$A \nsubseteq B, \quad B \nsubseteq A,$$

showing that R is not a total order.

A very important example of total order is the **inequality** on $\mathbb{Q}$.

Example 2.21:   Inequality is a total order

Consider the set of rationals $\mathbb{Q}$. The usual inequality defines a **binary relation** on $\mathbb{Q} \times \mathbb{Q}$, via

$$R := \{(x, y) \in \mathbb{Q} \times \mathbb{Q} : x \leq y\}.$$

Let us check that R is an **order relation**:

1. Reflexive: It holds, since $x \leq x$ for all $x \in \mathbb{Q}$,

2. Transitive: If $x \leq y$ and $y \leq z$ then $x \leq z$.

---

3. Antisymmetric: If $x \leq y$ and $y \leq x$ then $x = y$.

Finally, we halso have that R is a **total order** on $\mathbb{Q}$, since for all $x, y \in \mathbb{Q}$ we have

$$x \leq y \quad \text{or} \quad y \leq x.$$

Notation 2.22

If Ω is a set and R is a total order on Ω, we write

$$(x, y) \in R \iff x \leq y.$$

Therefore the symbol $\leq$ will always denote a total order relation.

2.6 Intervals

In this section we assume to have available the set $\mathbb{R}$ of **real numbers**, which we recall is an extension of $\mathbb{Q}$. We now introduce the concept of **interval**.

Definition 2.23

Let $a, b \in \mathbb{R}$ with $a < b$. We define the **open interval** (a, b) as the set

$$(a, b) := \{x \in \mathbb{R} : a < x < b\}.$$

We define the **close interval** $[a, b]$ as the set

$$[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}.$$

In general we also define the intervals

$$\begin{align}
[a, b) &:= \{x \in \mathbb{R} : a \leq x < b\}, & (2.17)\\
(a, b] &:= \{x \in \mathbb{R} : a \leq x \leq b\}, & (2.18)\\
(a, \infty) &:= \{x \in \mathbb{R} : x > a\}, & (2.19)\\
[a, \infty) &:= \{x \in \mathbb{R} : x \geq a\}, & (2.20)\\
(-\infty, b) &:= \{x \in \mathbb{R} : x < b\}, & (2.21)\\
(-\infty, b] &:= \{x \in \mathbb{R} : x \leq b\}. & (2.22)
\end{align}$$

Some of the above intervals are depicted in Figure 2.1, Figure 2.2, Figure 2.3, Figure 2.4 below.

Figure 2.1: Interval (a, b)

Figure 2.2: Interval $[a, b]$

2.7 Functions

Definition 2.24:   Functions

Let A and B be sets. A **function** from A to B is a rule which associates **at each** element $x \in A$ a **single** element $y \in B$. Notations:

- We write
$$f \; : \; A \to B$$
  to indicate such rule,
- For $x \in A$, we denote by
$$y \; := \; f(x) \in B$$
  the element associated with x by f.
- We will often denote the map f also by
$$x \mapsto f(x).$$

In addition:

- The set A is called the **domain** of f,
- The **range** of f is the set
$$\{y \in B : \; y = f(x) \text{ for some } x \in A\} \subseteq B.$$

Warning

We want to stress the importance of the first two sentences in Definition 2.24. Assume that $f : A \to B$ is a function. Then:

Figure 2.3: Interval (a, ∞)

Figure 2.4: Interval $(-\infty, b]$

- To each element $x \in A$ we can **only** associate **one** element $f(x) \in B$,
- Every element $x \in A$ has to be associated to an element $f(x) \in B$.

Example 2.25

Assume given the two sets

$$A = \{a_1, a_2\}, \quad B = \{b_1, b_2, b_3\}.$$

Let us see a few examples:

- Define $f : A \to B$ by setting

$$f(a_1) = b_1, \quad f(a_2) = b_1.$$

  In this way f is a function, with domain A and range

$$f(A) = \{b_1\} \subseteq B.$$

- Define $g : A \to B$ by setting

$$g(a_1) = b_2, \quad g(a_1) = b_3, \quad g(a_2) = b_3$$

  Then g is **NOT** a function, since the element a_1 has two elements associated.
- Define $h : A \to B$ by setting

$$h(a_1) = b_1.$$

  Then g is **NOT** a function, since the element a_2 has no element associated.

Example 2.26

Let us make two examples of functions on $\mathbb{R}$:

- Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = x^2.$$

  Note that the domain of f is given by $\mathbb{R}$, while the range is

$$f(\mathbb{R}) = [0, \infty).$$

- Define $g : \mathbb{R} \to \mathbb{R}$ as the logarithm:

$$g(x) = \log(x).$$

  This time the domain is $(0, \infty)$, while the range is $g(\mathbb{R}) = \mathbb{R}$.
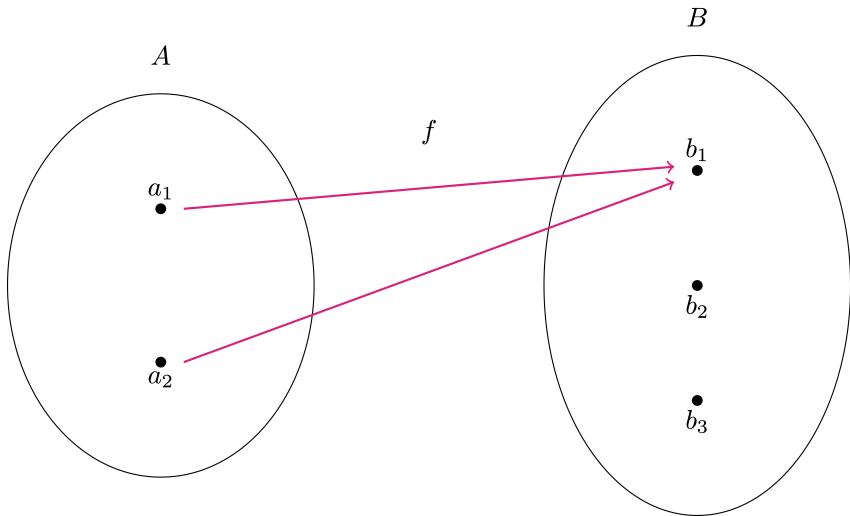
Figure 2.5: Schematic picture of the function f
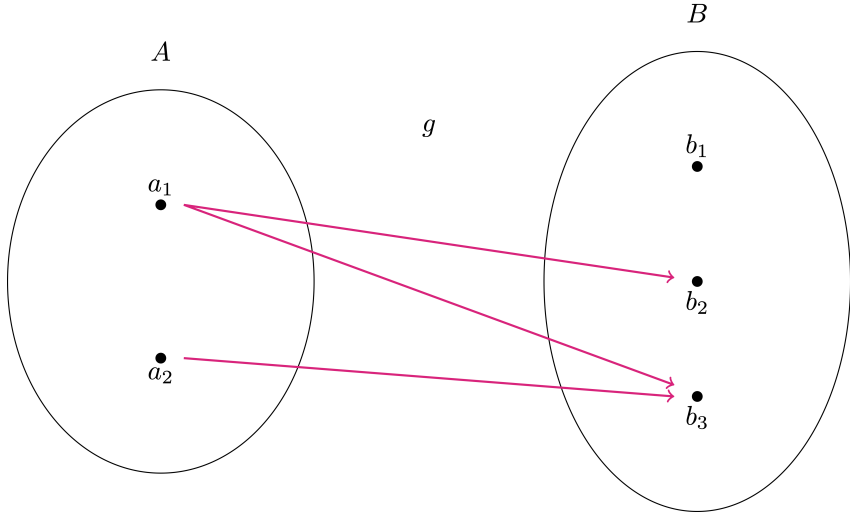
Figure 2.6: Schematic picture of the function g
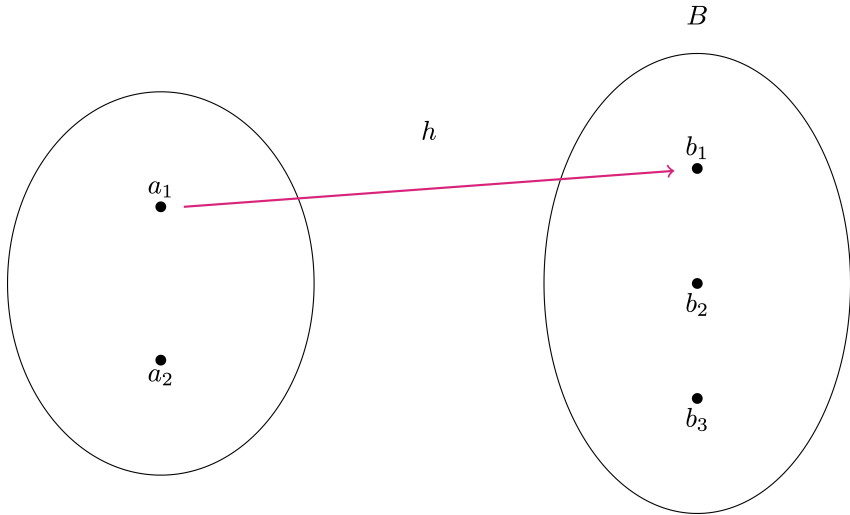
Figure 2.7: Schematic picture of the function h
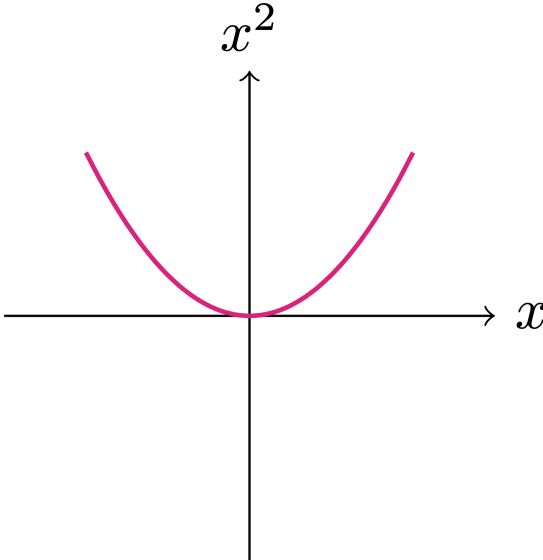
Figure 2.8: Plot of function $f(x) = x^2$

$$\log(x)$$
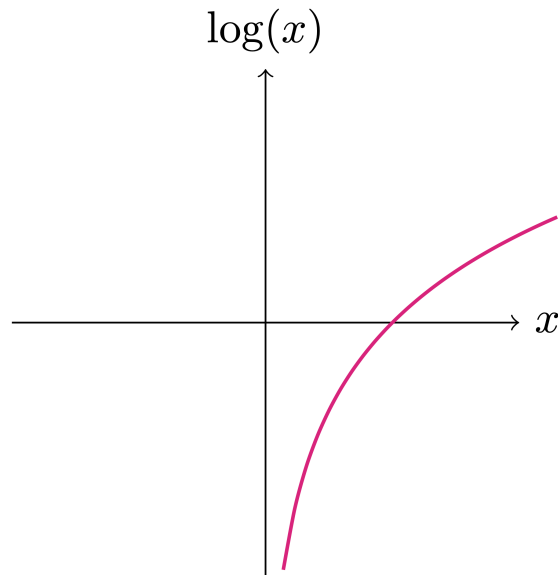
Figure 2.9: Plot of function $g(x) = \log(x)$

2.8 Absolute value or Modulus

In this section we assume to have available the set $\mathbb{R}$ of **real numbers**, which we recall is an extension of $\mathbb{Q}$.

Definition 2.27: Absolute value

For $x \in \mathbb{R}$ we define its **absolute value** as the quantity

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Example 2.28

By definition one has $|x| = x$ if $x \geq 0$. For example

$$|\pi| = \pi, \ \left|\sqrt{2}\right| = \sqrt{2}, \ |0| = 0.$$

Instead $|x| = -x$ if $x < 0$. For example

$$|-\pi| = \pi, \ \left|-\sqrt{2}\right| = \sqrt{2}, \ |-10| = 10.$$

Let us also make the following basic remark, whose proof will be left as an exercise.

Remark 2.29

For all $x \in \mathbb{R}$ one has
$$|x| \geq 0 \,.$$

Moreover
$$|x| = 0 \iff x = 0 \,.$$

Another basic remark (proof by exercise).

Remark 2.30

For all $x \in \mathbb{R}$ one has
$$|x| = |-x| \,.$$

We can use the definition of absolute value to define the **absolute value function**. This is the function

$$f \,:\, \mathbb{R} \to \mathbb{R}\,, \quad f(x) := |x| \,.$$

You might be familiar with the graph associated to f, as seen below.
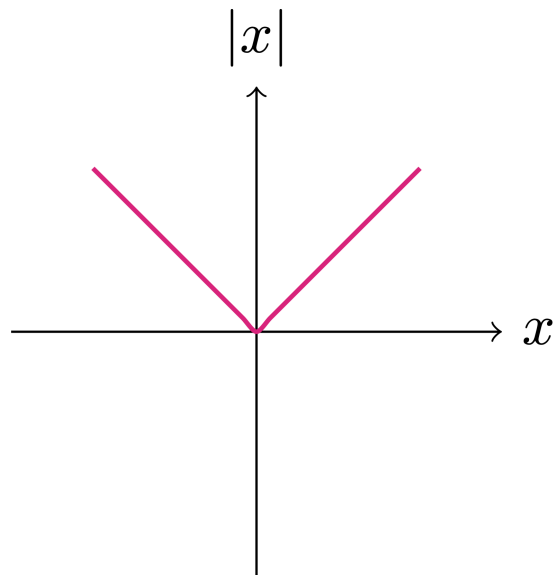
Figure 2.10: Plot of the absolute value function $f(x) = |x|$

It is also useful to understand the absolute value in a geometric way.

Remark 2.31: Geometric interpretation of $|x|$

A number $x \in \mathbb{R}$ can be represented with a point on the real line $\mathbb{R}$. The non-negative number $|x|$ represents the **distance** of x from the origin 0. Notice that this works for both positive and negative numbers x_1 and x_2 respectively, as shown in Figure 2.11 below.

$$|x_2| = -x_2 \qquad |x_1| = x_1$$
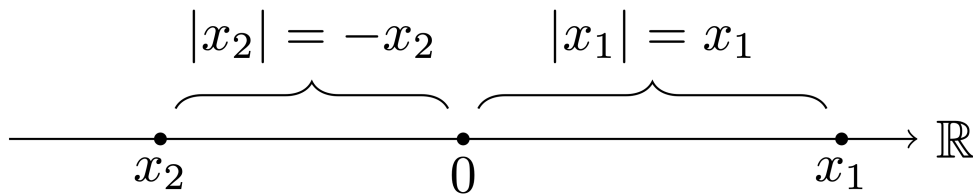
Figure 2.11: Geometric interpretation of $|x|$

Remark 2.32: Geometric interpretation of $|x - y|$

If $x, y \in \mathbb{R}$ then the number $|x - y|$ represents the distance between x and y on the real line, as shown in Figure 2.12 below. Note that by Remark 2.30 we have

$$|x - y| = |y - x|.$$
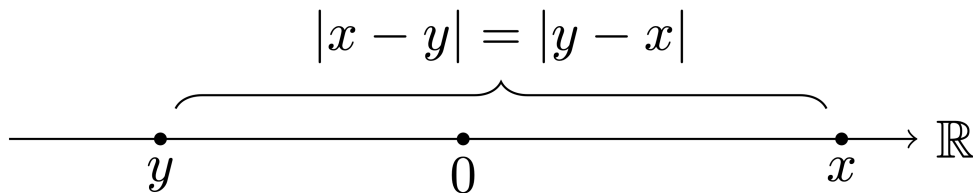
$$|x - y| = |y - x|$$

Figure 2.12: Geometric interpretation of $|x - y|$

In the next Lemma we show a fundamental equivalence regarding the absolute value.

Lemma 2.33

Let $x, y \in \mathbb{R}$. Then

$$|x| \leq y \iff -y \leq x \leq y.$$

The geometric meaning of the above statement is clear: the distance of x from the origin is less than y, in formulae

$$|x| \leq y,$$

if and only if x belongs to the interval $[-y, y]$, in formulae

$$-y \leq x \leq y.$$

A sketch of this explanation is seen in Figure 2.13 below.
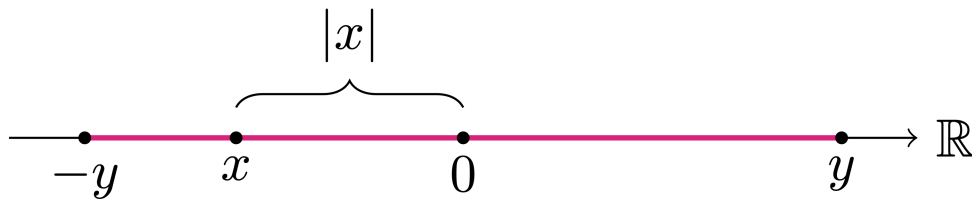
Figure 2.13: Geometric meaning of Lemma 2.33

Proof: Proof of Lemma 2.33

Step 1: First implication.
Suppose first that

$$|x| \leq y.$$ (2.23)

Recalling that the absolute value is non-negative, from (2.23) we deduce that $0 \leq |x| \leq y$. In particular it holds

$$y \geq 0.$$ (2.24)

We make separate arguments for the cases $x \geq 0$ and $x < 0$:

- Case 1: $x \geq 0$. From (2.23), (2.24) and from $x \geq 0$ we have

$$-y \leq 0 \leq x = |x| \leq y$$

   which shows

$$-y \leq x \leq y.$$

- Case 2: $x < 0$. From (2.23), (2.24) and from $x < 0$ we have

$$-y \leq 0 < -x = |x| \leq y$$

   which shows

$$-y \leq -x \leq y.$$

   Multiplying the above inequalities by -1 yields

$$-y \leq x \leq y.$$

Step 2: Second implication.
Suppose now that

$$-y \leq x \leq y.$$ (2.25)

We make separate arguments for the cases $x \geq 0$ and $x < 0$:

- Case 1: $x \geq 0$. Since $x \geq 0$, from (2.25) we get

$$|x| = x \leq y$$

showing that
$$|x| \leq y \, .$$

- Case 2: $x < 0$. Since $x < 0$, from (2.25) we have

$$-y \leq x = -|x| \, .$$

Multiplying the above inequality by -1 yields

$$|x| \leq y \, .$$

With the same arguments, just replacing $\leq$ with $<$, one can also show the following.

Lemma 2.34

Let $x, y \in \mathbb{R}$. Then
$$|x| < y \iff -y < x < y \, .$$

2.9 Triangle inequality

The triangle inequality relates the absolute value to the sum operation. It is a very important inequality, which we will use a lot in the future.

Theorem 2.35: Triangle inequality

For every $x, y \in \mathbb{R}$ we have
$$\big||x| - |y|\big| \leq |x + y| \leq |x| + |y| \, . \tag{2.26}$$

Before proceeding with the proof, let us discuss the geometric meaning of the triangle inequality.

Remark 2.36: Geometric meaning of triangle inequality

The notion of absolute value can be extended also to vectors in the plane. Suppose that x and y are two vectors in the plane, as in Figure 2.14 below. Then $|x|$ and $|y|$ can be interpreted as the **lengths** of these

---

vectors.

Using the rule of sum of vectors, we can draw $x + y$, as shown in Figure 2.15 below. From the picture it is evident that

$$|x + y| \leq |x| + |y|, \tag{2.27}$$

that is, *the length of each side of a triangle does not exceed the sum of the lengths of the two remaining sides.* Note that (2.27) is exactly the second inequality in (2.26). This is why (2.26) is called triangle inequality.
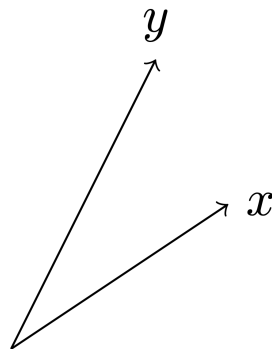
Figure 2.14: Vectors x and y
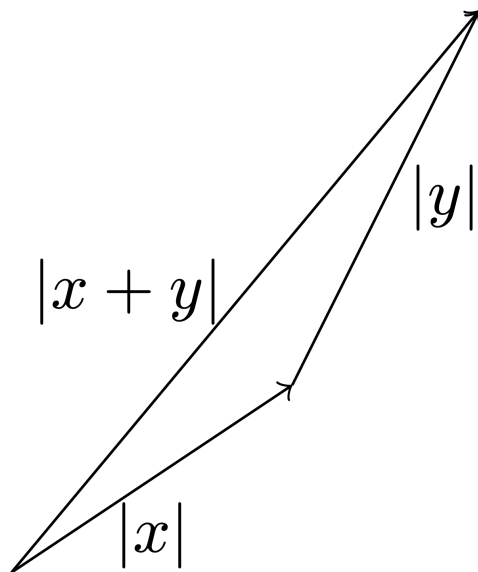
Figure 2.15: Summing the vectors x and y. The triangle inequality relates the length of $x + y$ to the length of x and y

Proof: Proof of Theorem 2.32

Assume that $x, y \in \mathbb{R}$. We prove the two inequalities in (2.26) individually.
Step 1. Proof of the second inequality in (2.26).
Trivially we have

$$|x| \leq |x| .$$

Therefore we can apply Lemma 2.33 and infer

$$-|x| \leq x \leq |x| . \qquad (2.28)$$

Similarly we have that $|y| \leq |y|$, and so Lemma 2.33 implies

$$-|y| \leq y \leq |y| . \qquad (2.29)$$

Summing (2.28) and (2.29) we get

$$-(|x| + |y|) \leq x + y \leq |x| + |y| .$$

We can now again apply Lemma 2.33 to get

$$|x + y| \leq |x| + |y| , \qquad (2.30)$$

which is the second inequality in (2.26).
Step 2. Proof of the first inequality in (2.26).
Note that the trivial identity

$$x = x + y - y$$

always holds. We then have

$$|x| = |x + y - y| \qquad (2.31)$$
$$= |(x + y) + (-y)| \qquad (2.32)$$
$$= |a + b| \qquad (2.33)$$

with $a = x + y$ and $b = -y$. We can now apply (2.30) to a and b to obtain

$$|x| = |a + b| \qquad (2.34)$$
$$\leq |a| + |b| \qquad (2.35)$$
$$= |x + y| + |-y| \qquad (2.36)$$
$$= |x + y| + |y| \qquad (2.37)$$

Therefore

$$|x| - |y| \leq |x + y| . \qquad (2.38)$$

We can now swap x and y in (2.38) to get

$$|y| - |x| \leq |x + y| .$$

By rearranging the above inequality we obtain

$$-|x + y| \leq |x| - |y| \, . \tag{2.39}$$

Putting together (2.38) and (2.39) yields

$$-|x + y| \leq |x| - |y| \leq |x + y| \, .$$

By Lemma 2.33 the above is equivalent to

$$\big||x| - |y|\big| \leq |x + y| \, ,$$

which is the first inequality in (2.26).

An immediate consequence of the triangle inequality are the following inequalities, which are left as an exercise.

Remark 2.37

For any $x, y \in \mathbb{R}$ it holds

$$\big||x| - |y|\big| \leq |x - y| \leq |x| + |y| \, .$$

Moreover for any $x, y, z \in \mathbb{R}$ it holds

$$|x - y| \leq |x - z| + |z - y| \, .$$

2.10 Proofs in Mathematics

In a mathematical proof one needs to show that

$$\alpha \implies \beta \tag{2.40}$$

where

- α is a given set of assumptions, or **Hypothesis**
- β is a conclusion, or **Thesis**

Proving (2.40) means convincing ourselves that β follows from α. Common strategies to prove (2.40) are:

1. **Contradiction**: Assume that the thesis is **false**, and hope to reach a contradiction: that is, prove that

$$\neg\beta \implies \text{contradiction}$$

where $\neg\beta$ is the **negation** of β.

For example we already proved by contradiction that

$$\text{Definition of } \mathbb{Q} \implies \sqrt{2} \notin \mathbb{Q},$$

In the above statement

$$\alpha = (\text{Definition of } \mathbb{Q}).$$
$$\beta = \left(\sqrt{2} \notin \mathbb{Q} \right).$$

Therefore

$$\neg \beta = \left(\sqrt{2} \in \mathbb{Q} \right).$$

2. **Direct**: Sometimes proofs will also need **direct** arguments, meaning that one need to show directly that (2.40) holds.

3. **Contrapositive**: The statement (2.40) is equivalent to

$$\neg \beta \implies \neg \alpha. \tag{2.41}$$

Thus, instead of proving (2.40), one could show (2.41). The statement (2.41) is called the **contrapositive** of (2.40).

Let us make an example.

Proposition 2.38

Two real numbers a, b are equal if and only if for every real number $\varepsilon > 0$ it follows that $|a - b| < \varepsilon$.

Before proceeding with the proof, note that the above stetement is just saying that:

Two numbers are equal if and only if they are **arbitrarily** close

By *arbitrarily close* we mean that they are *as close as you want the to be.*

Proof: of Proposition 2.38

Let us first rephrase the statement using mathematical symbols:

Let $a, b \in \mathbb{R}$. Then it holds:

$$a = b \iff |a - b| < \varepsilon, \ \forall \varepsilon > 0.$$

Setting

$$\alpha = (a = b) \tag{2.42}$$
$$\beta = (|a - b| < \varepsilon, \ \forall \varepsilon > 0) \tag{2.43}$$

the statement is equivalent to

$$\alpha \iff \beta \,.$$

To show the above, it is sufficient to show that

$$\alpha \implies \beta \quad \text{and} \quad \beta \implies \alpha \,.$$

Step 1. Proof that $\alpha \implies \beta$.
This proof can be carried out by a **direct** argument. Since we are assuming α, this means

$$a = b \,.$$

We want to see that β holds. Therefore fix an arbitrary $\varepsilon > 0$. This means that ε can be **any** positive number, as long as you fix it. Clearly

$$|a - b| = |0| = 0 < \varepsilon$$

since $a = b$, $|0| = 0$, and $\varepsilon > 0$. The above shows that

$$|a - b| < \varepsilon \,.$$

As $\varepsilon > 0$ was arbitrary, we have just proven that

$$|a - b| < \varepsilon \,, \quad \forall \, \varepsilon > 0 \,,$$

meaning that β holds and the proof is concluded.
Step 2. Proof that $\beta \implies \alpha$.
Let us prove this implication by showing the **contrapositive**

$$\neg \alpha \implies \neg \beta \,.$$

So let us assume $\neg \alpha$ is true. This means that

$$a \neq b \,.$$

We have to see that $\neg \beta$ holds. But $\neg \beta$ means that

$$\exists \, \varepsilon_0 > 0 \ \text{ s.t. } \ |a - b| \geq \varepsilon_0 \,.$$

The above is satisfied by choosing

$$\varepsilon_0 := |a - b| \,,$$

since $\varepsilon_0 > 0$ given that $a \neq b$.

2.11 Induction

Another technique for carrying out proofs is **induction**, which we take as an axiom.

Axiom 2.39: Principle of Induction

Let $S \subseteq \mathbb{N}$. Suppose that

1. We have $1 \in S$, and
2. Whenever $n \in S$, then $(n + 1) \in S$.

Then we have
$$S = \mathbb{N}.$$

Important

The above is an **axiom**, meaning that we do not prove it, but rather we just **assume it holds**.

Remark 2.40

It would be possible to prove the Principle of Induction starting from elementary axioms for $\mathbb{N}$, called the **Peano Axioms**, see the Wikipedia page.
However, in justifying basic principles of mathematics, one at some point needs to draw a line. This means that something which looks elementary needs to be assumed to hold, in order to have a starting point for proving deeper statements.
In the case of the Principle of Induction, the intuition is clear:

> The Principle of Induction is just describing the **domino effect**: *If one tile falls, then the next one will fall as well.* Therefore if the *first tile falls, all the tiles will fall.*

It seems reasonable to assume such evident principle.

The Principle of Induction can be used to prove statements which depend on some index $n \in \mathbb{N}$. Precisely, the following statement holds.

Corollary 2.41: Principle of Inducion - Alternative formulation

Let $\alpha(n)$ be a statement which depends on $n \in \mathbb{N}$. Suppose that

1. $\alpha(1)$ is true, and

---

2. Whenever $\alpha(n)$ is true, then $\alpha(n + 1)$ is true.

Then $\alpha(n)$ is true for all $n \in \mathbb{N}$.

Proof

Define the set
$$S := \{n \in \mathbb{N} \text{ s.t. } \alpha(n) \text{ is true}\}.$$
Then

1. We have $1 \in S$, since $\alpha(1)$ is true.
2. If $n \in S$ then $\alpha(n)$ is true. By assumption this implies that $\alpha(n + 1)$ is true. Therefore $(n + 1) \in S$.

Therefore S satisfies the assumptions of the Induction Principle and we conclude that

$$S = \mathbb{N}.$$

By definition this means that $\alpha(n)$ is true for all $n \in \mathbb{N}$.

Example 2.42: Formula for summing first n natural numbers

Using the Principle of Induction we can prove that

$$1 + 2 + 3 + \ldots + (n - 1) + n = \frac{n(n + 1)}{2} \tag{2.44}$$

holds for all $n \in \mathbb{N}$.
Proof. To be really precise, consider the statement

$$\alpha(n) := \text{ the above formula is true for } n.$$

In order to apply induction, we need to show that

1. $\alpha(1)$ is true,
2. If $\alpha(n)$ is true then $\alpha(n + 1)$ is true.

Let us proceed:

1. It is immediate to check that (2.44) holds for $n = 1$.

2. Suppose (2.44) holds for n. Then

$$1 + \ldots + n + (n+1) = \frac{n(n+1)}{2} + (n+1) \qquad (2.45)$$

$$= \frac{n(n+1) + 2(n+1)}{2} \qquad (2.46)$$

$$= \frac{(n+1)(n+2)}{n} \qquad (2.47)$$

where in the first equality we used that (2.44) holds for n. We then have

$$1 + \ldots + n + (n+1) = \frac{(n+1)(n+2)}{n}\,,$$

which shows that (2.44) holds for $n + 1$.

By the Principle of Induction we then conclude that $\alpha(n)$ is true for all $n \in \mathbb{N}$, which means that (2.44) holds for all $n \in \mathbb{N}$.

Example 2.43: Statements about sequences of numbers

Suppose you are given a collection of numbers

$$\{x_n \ \text{s.t.} \ n \in \mathbb{N}\}\,.$$

Such collection of numbers is called **sequence**. Assume that

$$x_1 := 1 \qquad (2.48)$$

$$x_{n+1} := \frac{x_n}{2} + 1\,. \qquad (2.49)$$

A sequence defined as above is called **recurrence sequence**. Using the above rule we can compute all the terms of x_n.

For example

$$x_2 = \frac{x_1}{2} + 1 = \frac{1}{2} + 1 = \frac{3}{2} \qquad (2.50)$$

$$x_3 = \frac{x_2}{2} + 1 = \frac{3}{4} + 1 = \frac{7}{4}\,. \qquad (2.51)$$

By computing these terms, we suspect that the sequence might be increasing, meaning that

$$x_{n+1} \geq x_n \qquad (2.52)$$

for all $n \in \mathbb{N}$.

Claim. (2.52) holds for all $n \in \mathbb{N}$.

Proof of Claim.

We argue by induction:

1. We have seen that $x_1 = 1$ and $x_2 = 3/2$. Thus

$$x_2 \geq x_1 \,.$$

2. Suppose now that

$$x_{n+1} \geq x_n \,. \qquad (2.53)$$

We need to prove that

$$x_{n+2} \geq x_{n+1} \,. \qquad (2.54)$$

Indeed, we can multiply the inequality (2.53) by $1/2$ and add 1 to get

$$\frac{x_{n+1}}{2} + 1 \geq \frac{x_n}{2} + 1 \,.$$

The above is equivalent, by definition, to (2.54).

Therefore the assumptions of the Induction Principle are satisfied, and (2.52) follows.

3 Real Numbers

In this chapter we introduce the system of Real Numbers $\mathbb{R}$ and study some of its properties.

3.1 Fields

In order to introduce $\mathbb{R}$, we need the concepts of **binary operation** and **field**. We proceed in a general setting, starting from a set K.

Definition 3.1: Binary operation

A binary operation on a set K is a function

$$\circ \; : \; K \times K \to K$$

which maps the ordered pair (x, y) into $x \circ y$.

Notation 3.2

There are two main binary operations we are interested in:

- **Addition**: denoted by $+$. The addition, or **sum** of $x, y \in K$ is denoted by

$$x + y.$$

- **Multiplication**: denoted by $\cdot$. The multiplication, or **product** of $x, y \in K$ is denoted by

$$x \cdot y \;\; \text{or} \;\; xy.$$

Example 3.3:   of binary operation

Let $K = \{0, 1\}$. We can for example define operations of sum and product on K according to the tables

$$
\begin{array}{c|cc}
+ & 0 & 1 \\
\hline
0 & 0 & 1 \\
1 & 1 & 0
\end{array}
\qquad
\begin{array}{c|cc}
\cdot & 0 & 1 \\
\hline
0 & 0 & 0 \\
1 & 0 & 1
\end{array}
$$

The above mean that

$$0 + 0 = 1 + 1 = 0, \quad 0 + 1 = 1 + 0 = 1,$$

$$0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0, \quad 1 \cdot 1 = 1.$$

This is just one option. Note that we could not have defined

$$1 + 1 = 2,$$

since $2 \notin K$.

Binary operations take ordered pairs of elements of K as input. Therefore the operation

$$x \circ y \circ z$$

does not make sense, since we do not know which one between

$$x \circ y \quad \text{or} \quad y \circ z$$

has to be performed first. Moreover the outcome of an operation depends on order:

$$x \circ y \neq y \circ x.$$

This motivates the following definition.

Definition 3.4

Let K be a set and $\circ : K \times K \to K$ be a binary operation on K. We say that:

1. $\circ$ is **commutative** if
$$x \circ y = y \circ x, \quad \forall\, x, y \in K$$

2. $\circ$ is **associative** if
$$(x \circ y) \circ z = x \circ (y \circ z), \quad \forall\, x, y, z \in K$$

3. An element $e \in K$ is called **neutral element** of $\circ$ if
$$x \circ e = e \circ x = x, \quad \forall\, x \in K$$

4. Let e be a neutral element of $\circ$ and let $x \in K$. An element $y \in K$ is called an **inverse** of x with respect to $\circ$ if
$$x \circ y = y \circ x = e.$$

Example 3.5

Let K with $+$ and $\cdot$ be as in Example 3.1. The sum satisfies:

- $+$ is commutative, since
$$0 + 1 = 1 + 0 = 0 \,.$$

- $+$ is associative, since for example
$$(0 + 1) + 1 = 1 + 1 = 0 \,, \quad 0 + (1 + 1) = 0 + 0 = 0 \,,$$

  and therefore
$$(0 + 1) + 1 = 0 + (1 + 1) \,.$$

  In general one can show that $+$ is associative by checking all the other permutations.
- The neutral element of $+$ is 0, since
$$0 + 0 = 0 \,, \quad 1 + 0 = 0 + 1 = 1 \,.$$

- Every element has an inverse. Indeed, the inverse of 0 is 0, since
$$0 + 0 = 0 \,,$$

  while the inverse of 1 is 1, since
$$1 + 1 = 1 + 1 = 0 \,.$$

The multiplication satisfies:

- $\cdot$ is commutative, since
$$1 \cdot 0 = 0 \cdot 1 = 0 \,.$$

- $\cdot$ is associative, since for example
$$(0 \cdot 1) \cdot 1 = 0 \cdot 1 = 0 \,, \quad 0 \cdot (1 \cdot 1) = 0 \cdot 1 = 0 \,,$$

  and therefore
$$(0 \cdot 1) \cdot 1 = 0 \cdot (1 \cdot 1) \,.$$

  By checking all the other permutations one can show that $\cdot$ is associative.
- The neutral element of $\cdot$ is 1, since
$$0 \cdot 1 = 1 \cdot 0 = 0 \,, \quad 1 \cdot 1 = 1 \,.$$

- The element 0 has no inverse, since
$$0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0 \,,$$

  and thus we never obtain the neutral element 1. The inverse of 1 is given by 1, since
$$1 \cdot 1 = 1 \,.$$

Example 3.6

Let $K = \{0, 1\}$ be a set with binary relation $\circ$ defined by the table

$$
\begin{array}{c|cc}
\circ & 0 & 1 \\
\hline
0 & 1 & 1 \\
1 & 0 & 0
\end{array}
$$

In this case $\circ$ is not commutative since

$$0 \circ 1 = 1, \quad 1 \circ 0 = 0$$

and therefore

$$0 \circ 1 \neq 1 \circ 0.$$

Moreover $\circ$ is not associative, since

$$(0 \circ 1) \circ 1 = 1 \circ 1 = 0,$$

while

$$0 \circ (1 \circ 1) = 0 \circ 0 = 1,$$

so that

$$(0 \circ 1) \circ 1 \neq 0 \circ (1 \circ 1).$$

We are ready to define fields.

Definition 3.7: Field

Let K be a set with binary operations of **addition**

$$+ \; : \; K \times K \to K, \quad (x, y) \mapsto x + y$$

and **multiplication**

$$\cdot \; : \; K \times K \to K, \quad (x, y) \mapsto x \cdot y = xy.$$

We call the triple $(K, +, \cdot)$ a **field** if:

1. The addition $+$ satisfies: $\forall\, x, y, z \in K$

   - (A1) **Commutativity and Associativity**:

     $$x + y = y + x$$

     $$(x + y) + z = x + (y + z)$$

   - (A2) **Additive Identity**: There exists a **neutral element** in K for $+$, which we call 0. It holds:

     $$x + 0 = 0 + x = x$$

- (A3) **Additive Inverse**: There exists an **inverse** of x with respect to $+$. We call this element the **additive inverse** of x and denote it by $-x$. It holds

$$x + (-x) = (-x) + x = 0$$

2. The multiplication $\cdot$ satisifes: $\forall\, x, y, z \in K$

- (M1) **Commutativity and Associativity**:

$$x \cdot y = y \cdot x$$

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

- (M2) **Multiplicative Identity**: There exists a **neutral element** in K for $\cdot$, which we call 1. It holds:

$$x \cdot 1 = 1 \cdot x = x$$

- (M3) **Multiplicative Inverse**: If $x \neq 0$ there exists an **inverse** of x with respect to $\cdot$. We call this element the **multiplicative inverse** of x and denote it by x^{-1}. It holds

$$x \cdot x^{-1} = x^{-1} \cdot x = 1$$

3. The operations $+$ and $\cdot$ are related by

- (AM) **Distributive Property**: $\forall\, x, y, z \in K$

$$x \cdot (y + z) = (x \cdot y) + (y \cdot z)\,.$$

Example 3.8

Let K with $+$ and $\cdot$ be as in Example 3.1. We can show that $(K, +, \cdot)$ is a field. Indeed we have already shown in Example 3.5 that:

- (A1) and (M1) hold,
- (A2) holds with neutral element 0,
- (M2) holds with neutral element 1,
- (A3) every element has an additive inverse, with

$$-0 = 0\,, \quad -1 = 1\,,$$

- (M3) every element which is not 0 a multiplicative inverse, with

$$1^{-1} = 1\,.$$

We are left to show the Distributive Property (AM). Indeed:

- (AM) For all $y, z \in K$ we have

$$0 \cdot (y + z) = 0, \quad (0 \cdot y) + (0 \cdot z) = 0 + 0 = 0,$$

and also

$$1 \cdot (y + z) = y + z, \quad (1 \cdot y) + (1 \cdot z) = y + z.$$

Thus (AM) holds.

Definition 3.9: Subtraction and division

Let $(K, +, \cdot)$ be a field. We define:

- **Subtraction** as the operation $-$ defined by

$$x - y := x + (-y), \quad \forall\, x, y \in K,$$

where $-y$ is the additive inverse of y.
- **Division** as the operation $/$ defined by

$$x/y := x \cdot y^{-1}, \quad \forall\, x, y \in K, \ y \neq 0,$$

where y^{-1} is the multiplicative inverse of y.

Proposition 3.10: Uniqueness of neutral elements and inverses

Let $(K, +, \cdot)$ be a field. Then

1. There is a unique element in K with the property of 0,
2. There is a unique element in K with the property of 1,
3. For all $x \in K$ there is a unique additive inverse $-x$,
4. For all $x \in K$, $x \neq 0$, there is a unique multiplicative inverse x^{-1}.

Proof

1. Suppose that $0 \in K$ and $\tilde{0} \in K$ are both neutral element of $+$, that is, they both satisfy (A2). Then

$$0 + \tilde{0} = 0$$

since $\tilde{0}$ is a neutral element for $+$. Moreover

$$\tilde{0} + 0 = \tilde{0}$$

since 0 is a neutral element for $+$. By commutativity of $+$, see property (A1), we have

$$0 = 0 + \tilde{0} = \tilde{0} + 0 = \tilde{0},$$

showing that $0 = \tilde{0}$. Hence the neutral element for $+$ is unique.

2. Exercise.

3. Let $x \in K$ and suppose that $y, \tilde{y} \in K$ are both additive inverses of x, that is, they both satisfy (A3). Therefore

$$x + y = 0$$

since y is an additive inverse of x and

$$x + \tilde{y} = 0$$

since $\tilde{y}$ is an additive inverse of x. Therefore we can use commutativity and associativity and of $+$, see property (A1), and the fact that 0 is the neutral element of $+$, to infer

$$\begin{aligned} y &= y + 0 = y + (x + \tilde{y}) \\ &= (y + x) + \tilde{y} = (x + y) + \tilde{y} \\ &= 0 + \tilde{y} = \tilde{y} , \end{aligned}$$

concluding that $y = \tilde{y}$. Thus there is a unique additive inverse of x, and

$$y = \tilde{y} = -x ,$$

with $-x$ the element from property (A3).

4. Exercise.

Using the properties of field we can also show that the usual properties of sum, subtraction, multiplication and division still hold in any field. We list such properties in the following proposition.

Proposition 3.11:   Properties of field operations

Let $(K, +, \cdot)$ be a field. Then for all $x, y, z \in K$,

- $x + y = x + z \implies y = z$
- $x \cdot y = x \cdot z$ and $x \neq 0 \implies y = z$
- $-0 = 0$
- $1^{-1} = 1$
- $x \cdot 0 = 0$
- $-1 \cdot x = -x$
- $-(-x) = x$
- $(x^{-1})^{-1} = x$ if $x \neq 0$
- $(x \cdot y)^{-1} = x^{-1} \cdot y^{-1}$

The above properties can be all proven with elementary use of the field properties (A1)-(A3), (M1)-(M3) and (AM). This is an exercise in patience, and is left to the reader.

Let us conclude with examining the sets of numbers introduced in Chapter 1.

Theorem 3.12

Consider the sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ with the usual operations $+$ and $\cdot$. We have:

- $(\mathbb{N}, +, \cdot)$ is **not a field**:
  It satisfies properties (A1), (A2), (M1), (M2), (AM) of fields. It is missing properties (A3) and (M3), the additive and multiplicative inverse properties, respectively.
- $(\mathbb{Z}, +, \cdot)$ is **not a field**:
  It satisfies properties (A1), (A2), (A3), (M1), (M2), (AM) of fields. Thus it is only missing (M3), the multiplicative inverse property.
- $(\mathbb{Q}, +, \cdot)$ **is a field**.

The proof is omitted.

3.2 Ordered fields

Definition 3.13

Let K be a set with binary operations $+$ and $\cdot$, and with an order relation $\leq$. We call $(K, +, \cdot, \leq)$ an **ordered field** if:

1. $(K, +, \cdot)$ is a field

2. There $\leq$ is of **total order** on K: $\forall\, x, y, z \in K$

   - (O1) **Reflexivity**:
     $$x \leq x$$

   - (O2) **Antisymmetry**:
     $$x \leq y \text{ and } y \leq x \implies x = y$$

   - (O3) **Transitivity**:
     $$x \leq y \text{ and } y \leq z \implies x = z$$

   - (O4) **Total order**:
     $$x \leq y \text{ or } y \leq x$$

3. The operations $+$ and $\cdot$, and the total order $\leq$, are related by the following properties: $\forall x, y, z \in K$

   - (AM) **Distributive**: Relates addition and multiplication via
     $$x \cdot (y + z) = x \cdot y + x \cdot z$$

   - (AO) Relates addition and order with the requirement:
     $$x \leq y \implies x + z \leq y + z$$

   - (MO) Relates multiplication and order with the requirement:
     $$x \geq 0, \, y \geq 0 \implies x \cdot y \geq 0$$

Example 3.14

$(\mathbb{Q}, +, \cdot, \leq)$ is an **ordered field**.

3.3 Cut Property

We have just introduced the notion of **field**, and noted that the set of rational numbers with the usual operations

$$(\mathbb{Q}, +, \cdot)$$

is a **field**.

We now need to address the key issue we proved in Chapter 1, that is, that

$$\sqrt{2} \notin \mathbb{Q}.$$

This means that $\mathbb{Q}$ has **gaps**, and cannot be represented as a **continuous** line. The rigorous definition of **lack of gaps** needs the concept of **cut** of a set.

Definition 3.15: Partition of a set

Let S be a non-empty set. The pair (A, B) is a **partition** of S if

$$A, B \subseteq S, \quad A \neq \emptyset, \quad B \neq \emptyset,$$

and

$$S = A \cup B, \quad A \cap B = \emptyset.$$
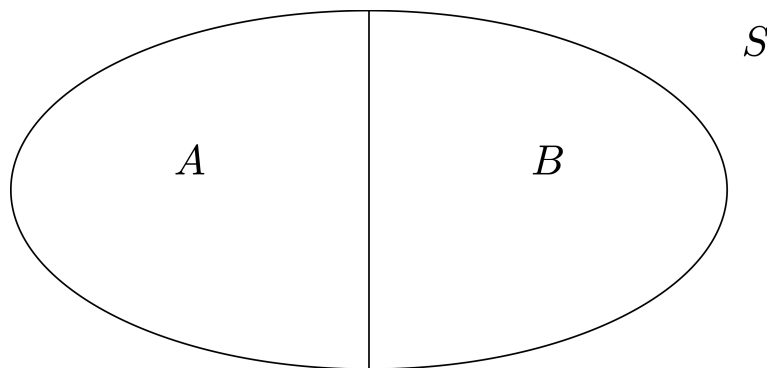
Figure 3.1: Schematic picture of a partition (A, B) of the set K.

Definition 3.16: Cut of a set

Let S be a non-empty set with a total order relation $\leq$. The pair (A, B) is a **cut** of S if

1. (A, B) is a **partition** of S,
2. We have
$$a \leq b, \quad \forall a \in A, \ \forall b \in B.$$

The **cut** of a set is often called **Dedekind cut**, named after **Richard Dedekind**, who used cuts to give an explicit construction of the real numbers $\mathbb{R}$, see Wikipedia page.

Definition 3.17: Cut property

Let S be a non-empty set with a total order relation $\leq$. We say that S has the **cut property** if for every cut (A, B) of S there exists some $s \in S$ such that

$$a \leq s \leq b, \quad \forall a \in A, \ \forall b \in B.$$

We call s the **separator** of the cut (A, B).

Example 3.18

Let $S = \mathbb{Q}$ and consider the sets

$$A = (-\infty, s] \cap \mathbb{Q}, \quad B = (s, \infty) \cap \mathbb{Q}.$$

for some $s \in \mathbb{Q}$. Then the pair (A, B) is a cut of $\mathbb{Q}$, and s is the separator.

$$A = (-\infty, s] \cap \mathbb{Q} \qquad\qquad B = (s, \infty) \cap \mathbb{Q}$$

$$s$$
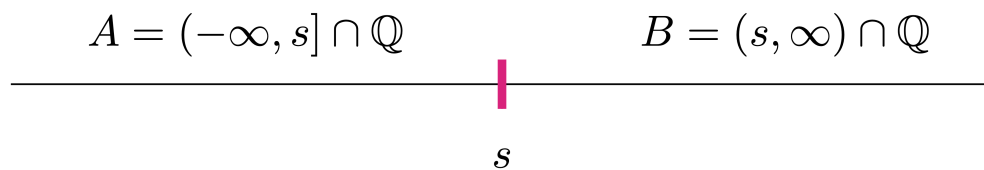
Figure 3.2: (A, B) is a cut of $\mathbb{Q}$ with separator s.

Question 3.19

Do all ordered fields have the Cut Property? Does $\mathbb{Q}$ have the Cut Property?

The answer to the above question is **NO**. For example the pair

$$A = (-\infty, \sqrt{2}) \cap \mathbb{Q}, \quad B = (\sqrt{2}, \infty) \cap \mathbb{Q}. \tag{3.1}$$

is a cut of $\mathbb{Q}$, since $\sqrt{2} \notin \mathbb{Q}$. However what is the separator? It should be $s = \sqrt{2}$, given that clearly

$$a \leq \sqrt{2} \leq b, \quad \forall\, a \in A, \ \forall\, b \in B.$$

However $\sqrt{2} \notin \mathbb{Q}$, so we are **NOT ALLOWED** to take it as separator. Indeed, we can show that (A, B) defined as in (3.1) has no separator.

$$A = (-\infty, \sqrt{2}) \cap \mathbb{Q} \qquad\qquad B = (\sqrt{2}, \infty) \cap \mathbb{Q}$$

$$\sqrt{2} \notin \mathbb{Q}$$
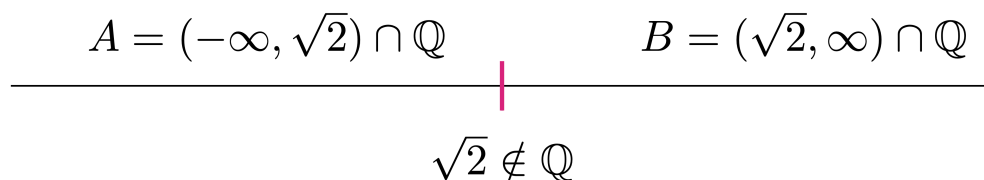
Figure 3.3: (A, B) is a cut of $\mathbb{Q}$ which has no separator

Theorem 3.20: $\mathbb{Q}$ does not have the cut property.

$\mathbb{Q}$ does not have the cut property. More explicitly, there exist a cut (A, B) of $\mathbb{Q}$ which has no separator.

Remark 3.21:   Ideas for the proof of Theorem 3.20

Before proceeding with the proof, let us summarize the ideas behind it:
We will consider the cut (A, B) in (3.1). We then assume by contradiction that (A, B) admits a separator $L \in \mathbb{Q}$, so that

$$a \leq L \leq b, \quad \forall\, a \in A, \ \forall\, b \in B. \tag{3.2}$$

Since (A, B) is a partition of $\mathbb{Q}$, then either $L \in A$ or $L \in B$. These will both lead to a contradiction:

- If $L \in A$, by definition of A we have

$$L < \sqrt{2}.$$

  We want to contradict the fact that L is a separator for the cut (A, B). The idea is that $\sqrt{2} \notin \mathbb{Q}$, and therefore it is possible to find a rational number $\tilde{L}$ such that

$$L < \tilde{L} < \sqrt{2}.$$

  How do we find such $\tilde{L}$ in practice? We look for a number $\tilde{L}_n$ of the form

$$\tilde{L}_n = 1 + \frac{1}{n}$$

  for some $n \in \mathbb{N}$ to be suitably chosen later. Clearly $\tilde{L}_n \in \mathbb{Q}$ and

$$L < \tilde{L}_n$$

  for all $n \in \mathbb{N}$. We need to prove that we can find $n_0 \in \mathbb{N}$ such that

$$L < \tilde{L}_{n_0} < \sqrt{2}. \tag{3.3}$$

  This is indeed possible: There exists $n_0 \in \mathbb{N}$ such that (3.3) holds. From (3.3) we see that $\tilde{L}_{n_0} \in A$. Since L is a separator, from (3.2) we obtain

$$\tilde{L}_{n_0} \leq L,$$

  which contradicts (3.3).

- If $L \in B$, by definition of B we have

$$\sqrt{2} < L.$$

  The idea is the same as above: Since $\sqrt{2} \notin \mathbb{Q}$, we can find $\tilde{L} \in \mathbb{Q}$ such that

$$\sqrt{2} < \tilde{L} < L.$$

Since we want $\tilde{L}$ to be a rational number smaller than L, we look for $\tilde{L}$ of the form

$$\tilde{L}_n := L - \frac{1}{n},$$

for a suitable $n \in \mathbb{N}$. This satisfies $\tilde{L}_n \in \mathbb{Q}$ and

$$\tilde{L}_n < L,$$

for all $n \in \mathbb{N}$. We then find $n_0 \in \mathbb{N}$ such that

$$\sqrt{2} < \tilde{L}_{n_0} < L. \qquad (3.4)$$

The above shows that $\tilde{L}_{n_0} \in B$. As L is a separator, we find that

$$L \leq \tilde{L}_{n_0},$$

which contradicts (3.4).

Therefore, both cases $L \in A$ or $L \in B$ lead to a contradiction. Since these are all the possibilities, we conclude that the cut (A, B) has no separator in $\mathbb{Q}$.

Time to make the ideas in the above remark rigorous.

Proof: Proof of Theorem 3.20

Let A and B be the sets defined in (3.1). It is useful to rewrite A and B in the form

$$A = A_1 \cup A_2,$$

where

$$A_1 = \{q \in \mathbb{Q} : q < 0\},$$
$$A_2 = \{q \in \mathbb{Q} : q \geq 0, q^2 < 2\},$$

and

$$B = \{q \in \mathbb{Q} : q > 0, q^2 > 2\}.$$

Step 1. (A, B) is a cut of $\mathbb{Q}$:
We need to prove the following:

1. (A, B) is a partition of $\mathbb{Q}$. This is because $A, B \subseteq \mathbb{Q}$ with $A \neq \emptyset$ and $B \neq \emptyset$. Moreover $A \cap B = \emptyset$ and

$$A \cup B = \mathbb{Q},$$

given that $\sqrt{2} \notin \mathbb{Q}$, and so there is no element $q \in \mathbb{Q}$ such that $q^2 = 2$.

2. It holds
$$a \leq b, \quad \forall a \in A, \ \forall b \in B.$$

Indeed, suppose that $a \in A$ and $b \in B$. We have two cases:

- $a \in A_1$: Therefore $a < 0$. In particular

$$a < 0 < b,$$

given that $b > 0$ for all $b \in B$. Thus $a < b$.
- $a \in A_2$: Therefore $a \geq 0$ and $a^2 < 2$. In particular

$$a^2 < 2 < b^2,$$

since $b^2 > 2$ for all $b \in B$. In particular

$$a^2 < b^2.$$

Since $b > 0$ for all $b \in B$, from the above inequality we infer $a < b$, concluding.

Step 2. (A, B) has no separator:
Suppose by contradiction that (A, B) admits a separator

$$L \in \mathbb{Q}.$$

By definition this means
$$a \leq L \leq b, \quad \forall a \in A, \ \forall b \in B. \tag{3.5}$$

Since
$$L \in \mathbb{Q}, \quad \mathbb{Q} = A \cup B, \quad A \cap B = \varnothing,$$

then either $L \in A$ or $L \in B$. We will see that both these possibilities lead to a contradiction:
Case 1: $L \in A$.
By (3.5) we know that
$$a \leq L, \quad \forall a \in A. \tag{3.6}$$

In particular the above implies
$$L \geq 0 \tag{3.7}$$

since $0 \in A$. Therefore we must have $L \in A_2$, that is,

$$L \geq 0 \ \text{ and } \ L^2 < 2. \tag{3.8}$$

Set
$$\tilde{L} := L + \frac{1}{n}$$

for $n \in \mathbb{N}, n \neq 0$ to be chosen later. Clearly we have

$$\tilde{L} \in \mathbb{Q} \ \text{ and } \ L < \tilde{L}. \tag{3.9}$$

From (3.7) and (3.9) we have also

$$\tilde{L} > 0 \,. \tag{3.10}$$

We now want to show that there is a choice of n such that $\tilde{L}^2 < 2$, which will lead to a contradiction. Indeed, we can estimate

$$\begin{aligned}
\tilde{L}^2 &= \left(L + \frac{1}{n}\right)^2 \\
&= L^2 + \frac{1}{n^2} + 2\frac{L}{n} \\
&< L^2 + \frac{1}{n} + 2\frac{L}{n} \qquad \left(\text{using } \frac{1}{n} < \frac{1}{n^2}\right) \\
&= L^2 + \frac{2L+1}{n} \,.
\end{aligned}$$

If we now impose that

$$L^2 + \frac{2L+1}{n} < 2 \,,$$

we can rearrange the above and obtain

$$n(2 - L^2) > 2L + 1 \,.$$

Now note that $L^2 < 2$ by assumption (3.8). Thus we can divived by $(2 - L^2)$ and obtain

$$n > \frac{2L+1}{2 - L^2} \,.$$

Therefore we have just shown that

$$n > \frac{2L+1}{2 - L^2} \quad \implies \quad \tilde{L}^2 < 2 \,.$$

Together with (3.10) this implies $\tilde{L} \in A$. Therefore we have

$$\tilde{L} \le L$$

by (3.6). On the other hand it also holds

$$\tilde{L} > L$$

by (3.9), and therefore we have a contradiction. Thus $L \notin A$.
Case 2: $L \in B$.
As $L \in B$, we have by definition

$$L > 0 \,, \quad L^2 > 2 \,. \tag{3.11}$$

Moreover since L is a separator, see (3.5), in particular

$$L \le b \,, \quad \forall\, b \in B \,. \tag{3.12}$$

Define now
$$\tilde{L} := L - \frac{1}{n}$$

with $n \in \mathbb{N}$, $n \neq 0$ to be chosen later. Clearly we have

$$\tilde{L} \in \mathbb{Q}, \quad \tilde{L} < L. \tag{3.13}$$

We now show that n can be chosen so that $\tilde{L} \in B$. Indeed

$$
\begin{aligned}
\tilde{L}^2 &= \left(L - \frac{1}{n} \right)^2 \\
&= L^2 + \frac{1}{n^2} - 2\frac{L}{n} \\
&> L^2 - \frac{1}{n^2} - 2\frac{L}{n} \qquad \left(\text{using } \frac{1}{n^2} > -\frac{1}{n^2} \right) \\
&> L^2 - \frac{1}{n} - 2\frac{L}{n} \qquad \left(\text{using } -\frac{1}{n^2} > -\frac{1}{n} \right) \\
&= L^2 - \frac{1 + 2L}{n}.
\end{aligned}
$$

Now we impose

$$L^2 - \frac{1 + 2L}{n} > 2$$

which is equivalent to

$$n(L^2 - 2) > 1 + 2L.$$

Since we are assuming $L \in B$, then $L^2 > 2$, see (3.11). Therefore we can divide by $(L^2 - 2)$ and get

$$n > \frac{1 + 2L}{L^2 - 2}.$$

In total, we have just shown that

$$n > \frac{1 + 2L}{L^2 - 2} \quad \implies \quad \tilde{L}^2 > 2,$$

proving that $\tilde{L} \in B$. Therefore by (3.12) we get

$$L \leq \tilde{L}.$$

This contradicts (3.13).
Conclusion:
We have seen that assuming that (A, B) has a separator $L \in \mathbb{Q}$ leads to a contradiction. Thus the cut (A, B) has no separator.

> **Remark 3.22**
>
> The above proof can be summarized by saying that the set
>
> $$A = (-\infty, \sqrt{2}) \cap \mathbb{Q}$$
>
> does not admit a **largest** element in $\mathbb{Q}$, and that the set
>
> $$B = (\sqrt{2}, \infty) \cap \mathbb{Q}$$
>
> does not admit a **lowest element** in $\mathbb{Q}$. We will clarify this remark in the next section.

3.4 Supremum and infimum

A crucial definition in Analysis is the one of supremum or infimum of a set. This is also another way of studying the **gaps** of $\mathbb{Q}$.

> **Example 3.23:**   Intuition about supremum and infimum
>
> Consider the set
> $$A = [0, 1) \cap \mathbb{Q}.$$
> Intuitively, we understand that A is bounded, i.e. not infinite. We also see that
>
> - 0 is the lowest element of A
> - 1 is the highest element of A
>
> However we see that $0 \in A$ while $1 \notin A$. We will see that
>
> - 0 can be defined as the **infimum** and **minimum** of A.
> - 1 can be defined as the **supremum**, but not **maximum**, of A.

3.4.1 Upper bound, supremum, maximum

We start by defining the supremum. First we need the notion of upper bound of a set.

> **Definition 3.24:**   Upper bound and bounded above
>
> Let $(K, +, \cdot, \leq)$ be an ordered field and $A \subseteq K$:
>
>   1. We say that $b \in K$ is an **upper bound** for A if
> $$a \leq b, \quad \forall\, a \in A.$$
>   2. We say that A is **bounded above** if there exists and upper bound $b \in K$ for A.

Definition 3.25: Supremum

Let $(K, +, \cdot, \leq)$ be an ordered field and $A \subseteq K$. A number $s \in K$ is called **least upper bound** or **supremum** of A if:

  i. s is an upper bound for A,
  ii. s is the smallest upper bound of A, that is,

$$\text{If } b \in K \text{ is upper bound for } A \text{ then } s \leq b.$$

Notation 3.26

We will almost always prefer the name supremum to least upper bound. For $A \subseteq K$ the supremum is denoted by

$$s := \sup A.$$

Remark 3.27

Note that if a set $A \subseteq K$ in **NOT** bounded above, then the supremum does not exist, as there are no upper bounds of A.

Proposition 3.28

Let $(K, +, \cdot, \leq)$ be an ordered field and $A \subseteq K$. If

$$\sup A$$

exists, then it is unique.

Proof

Suppose there exist $s_1, s_2 \in K$ such that

$$s_1 = \sup A, \ \ s_2 = \sup A.$$

Then:

  - Since $s_2 = \sup A$, in particular s_2 is an upper bound for A. Since $s_1 = \sup A$ then s_1 is the lowest upper bound. Thus we get
$$s_1 \leq s_2.$$

- Exchanging the roles s_1 and s_2 in the above reasoning we also get

$$s_2 \leq s_1\,.$$

This shows $s_1 = s_2$.

Warning

In general:

- A set can have infinite upper bounds,
- The supremum does not belong to the set.

For example
$$A = [0, 1) \cap \mathbb{Q}$$
has for upper bounds all the numbers $b \in \mathbb{Q}$ with $b > 1$. Moreover one can show that

$$\sup A = 1\,,$$

and so
$$\sup A \notin A\,.$$

Warning

The supremum does not exist in general. For example let

$$A = [0, \sqrt{2}) \cap \mathbb{Q}\,.$$

We will show that $\sup A$ does not exist in $\mathbb{Q}$. Indeed we will have that

$$\sup A = \sqrt{2} \in \mathbb{R}\,.$$
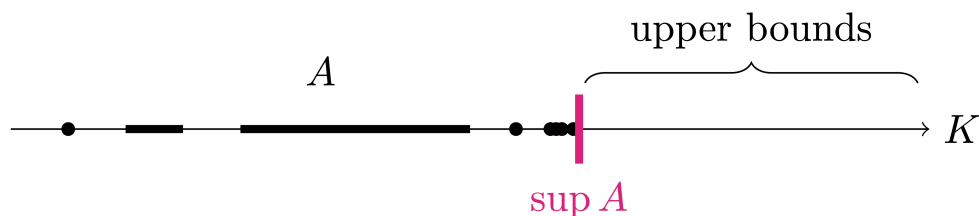
Figure 3.4: Supremum and upper bounds of a set A in the field K

Definition 3.29: Maximum

Let $(K, +, \cdot, \leq)$ be an ordered field and $A \subseteq K$. A number $M \in K$ is called the **maximum** of A if:

$$M \in A \text{ and } a \leq M, \forall a \in A.$$

We denote the maximum by
$$M = \max A.$$

Proposition 3.30

Let $(K, +, \cdot, \leq)$ be an ordered field and $A \subseteq K$. If the maximum of A exists, then also the supremum exists, and
$$\sup A = \max A.$$

Proof

Let
$$M = \max A.$$

Then:

- By definition we have $M \in A$ and
$$a \leq M, \quad \forall a \in A.$$
In particular the above tells us that M is an upper bound of A.
- We claim that M is the least upper bound. Indeed, suppose b is an upper bound of A, that is,

$$a \leq b, \quad \forall a \in A.$$

In particular, since $M \in A$, by the above condition we have

$$M \leq b.$$

Therefore M is the least upper bound of A, meaning that $M = \sup A$.

Warning

The converse of the above statement is not true: In general the sup might exist while the max does not.

For example
$$A = [0, 1) \cap \mathbb{Q}$$

is such that
$$\sup A = 1$$

but max A does not exist. Instead for the set
$$B = [0, 1] \cap \mathbb{Q}$$

we have that
$$\max A = \sup A = 1 \,.$$

3.4.2 Lower bound, infimum, minimum

We now introduce the definitions of lower bound, infimum, minimum. These are the counterpart of upper bound, supremum and maximum, respectively.

Definition 3.31: Upper bound, bounded below, infimum, minimum

Let $(K, +, \cdot, \leq)$ be an ordered field and $A \subseteq K$:

1. We say that $l \in K$ is a **lower bound** for A if
$$l \leq a \,, \quad \forall \, a \in A \,.$$

2. We say that A is **bounded below** if there exists a lower bound $l \in K$ for A.

3. We say that $i \in K$ is the **greatest lower bound** or **infimum** of A if:
   - i is a lower bound for A,
   - i is the largest lower bound of A, that is,

   $$\text{If } l \in K \text{ is a lower bound for } A \text{ then } l \leq i \,.$$

   If it exists, the infimum is denoted by
   $$i = \inf A \,.$$

4. We say that $m \in K$ is the **minimum** of A if:
   $$m \in A \text{ and } m \leq a \,, \forall a \in A \,.$$

   If it exists, we denote the minimum by
   $$m = \min A \,.$$
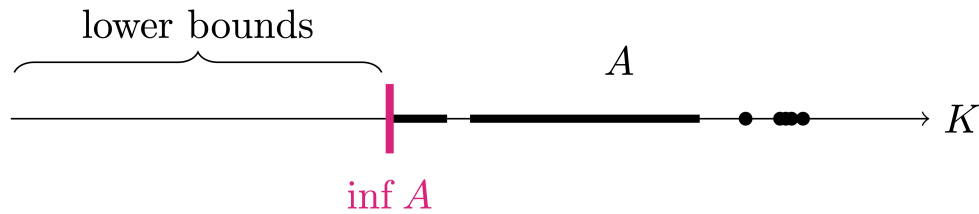
Figure 3.5: Infimum and lower bounds of a set A in the field K

Proposition 3.32

Let $(K, +, \cdot, \leq)$ be an ordered field and $A \subseteq K$:

1. If $\inf A$ exists, then it is unique.
2. If the minimum of A exists, then also the infimum exists, and

$$\inf A = \min A.$$

The proof uses similar arguments to the one employed in the previous section, and is left to the reader as an exercise.

Warning

We have

- A set can have infinite lower bounds,
- The infimum does not belong to the set.

For example

$$A = (0, 1) \cap \mathbb{Q}$$

has for lower bounds all the numbers $b \in \mathbb{Q}$ with $b < 1$. Moreover we will show that

$$\inf A = 0,$$

and so

$$\inf A \notin A.$$

Warning

The infimum does not exist in general. For example let

$$A = (\sqrt{2}, 5] \cap \mathbb{Q}.$$

We will show that inf A does not exist in $\mathbb{Q}$. Indeed we will have that

$$\inf A = \sqrt{2} \in \mathbb{R}.$$

Warning

In general the inf might exist while the min does not. For example

$$A = (0, 1) \cap \mathbb{Q}$$

is such that

$$\inf A = 0$$

but min A does not exist. Instead for the set

$$B = [0, 1] \cap \mathbb{Q}$$

we have that

$$\inf A = \min A = 0.$$

Proposition 3.33

Let $(K, +, \cdot, \leq)$ be an ordered field and $A \subseteq K$. If inf A and sup A exist, then

$$\inf A \leq a \leq \sup A, \quad \forall a \in A.$$

The proof is simple, and is left as an exercise. We now have a complete picture about supremum and infimum, see figure below.
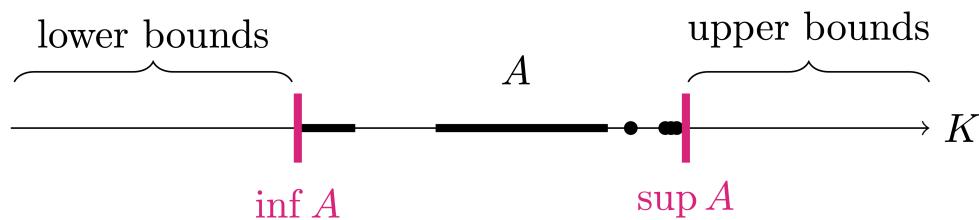
Figure 3.6: Supremum, upper bounds, infimum and lower bounds of a set A in K

We conclude with another simple proposition. The proof is again left to the reader.

Proposition 3.34:   Relationship between sup and inf

Let $(K, +, \cdot, \leq)$ be an ordered field and $A \subseteq K$. Define

$$-A := \{-a \, : \, a \in A\} \,.$$

It holds:

- If sup A exists, then inf A exists and

$$\inf(-A) = -\sup A \,.$$

- If inf A exists, then sup A exists and

$$\sup(-A) = -\inf A \,.$$

3.5 Completeness

We have introduced the concepts of supremum and infimum on an ordered field K.

Question 3.35

Suppose $(K, +, \cdot, \leq)$ is an ordered field, and that $A \subseteq K$ is non-empty and bounded above. Does

$$\sup A$$

always exist?

The answer to the above question is **NO**. Like we did with the Cut Property, the counterexample can be found in the set of rational numbers $\mathbb{Q}$. A set bounded above for which the supremum does nor exist is, for example,

$$A = [0, \sqrt{2}) \cap \mathbb{Q} \,. \tag{3.14}$$

Theorem 3.36

There exists a set $A \subseteq \mathbb{Q}$ such that

- A is non-empty,
- A is bounded above,
- sup A does not exist in $\mathbb{Q}$.

The proof uses the same ideas we used for showing that $\mathbb{Q}$ does not have the Cut Property.

Proof

Define the set A as in (3.14). Equivalently, this can be written as

$$A = \{q \in \mathbb{Q} : q \geq 0, \, q^2 < 2\}.$$

Step 1. A is bounded above.
Take $b := 9$. Then b is an upper bound for A. Indeed by definition

$$q^2 < 2, \, q \geq 0, \quad \forall q \in A.$$

Therefore

$$q^2 < 2 < 9 \implies q^2 < 9 \implies q < 3 = b.$$

Step 2. $\sup A$ does not exist.
Assume by contradiction that

$$s = \sup A \in \mathbb{Q}$$

exists. By definition it holds

$$s \geq q, \quad \forall q \in A \tag{3.15}$$

$$b \geq q, \forall q \in A \implies s \leq b \tag{3.16}$$

There are two possibilities: $s \in A$ or $s \notin A$:

- *Case 1. $s \in A$.*
  If $s \in A$ by definition

$$s \geq 0, \quad s^2 < 2. \tag{3.17}$$

  Define

$$\tilde{s} := s + \frac{1}{n}$$

with $n \in \mathbb{N}, n \neq 0$ to be chosen later. Then

$$\begin{aligned}
\tilde{s}^2 &= \left(s + \frac{1}{n}\right)^2 \\
&= s^2 + \frac{1}{n^2} + 2\frac{s}{n} \\
&< s^2 + \frac{1}{n} + 2\frac{s}{n} \quad \left(\text{using } \frac{1}{n} < \frac{1}{n^2}\right) \\
&= s^2 + \frac{2s+1}{n}.
\end{aligned}$$

  If we now impose that

$$s^2 + \frac{2s+1}{n} < 2,$$

we can rearrange the above and obtain

$$n(2 - s^2) > 2s + 1.$$

Now note that $s^2 < 2$ by assumption (3.17). Thus we can divide by $(2 - s^2)$ and obtain

$$n > \frac{2s + 1}{2 - s^2} \,.$$

To summarize, we have just shown that

$$n > \frac{2s + 1}{2 - s^2} \implies \tilde{s}^2 < 2 \,.$$

Moreover $\tilde{s} := (s + 1/n) \in \mathbb{Q}$. Therefore

$$\tilde{s} \in A \,.$$

Since $s = \sup A$, we then have

$$\tilde{s} \leq s \,.$$

However

$$\tilde{s} := s + \frac{1}{n} > s \,,$$

yielding a contradiction. Thus $s \in A$ is not possible.

- *Case 2. $s \notin A$.*
  If $s \notin A$, by the fact that $s = \sup A$ and by definition of A we get

$$s > 0 \,, \quad s^2 > 2 \,. \tag{3.18}$$

Define

$$\tilde{s} := s - \frac{1}{n} \,.$$

We have

$$\tilde{s}^2 = \left(s - \frac{1}{n} \right)^2$$
$$= s^2 + \frac{1}{n^2} - 2\frac{s}{n}$$
$$> s^2 - \frac{1}{n^2} - 2\frac{s}{n} \quad \left(\text{using } \frac{1}{n^2} > -\frac{1}{n^2} \right)$$
$$> s^2 - \frac{1}{n} - 2\frac{s}{n} \quad \left(\text{using } -\frac{1}{n^2} > -\frac{1}{n} \right)$$
$$= s^2 - \frac{1 + 2s}{n} \,.$$

Now we impose

$$s^2 - \frac{1 + 2s}{n} > 2$$

which is equivalent to

$$n(s^2 - 2) > 1 + 2s \,.$$

By (3.18) we have $s^2 > 2$. Therefore we can divide by $(s^2 - 2)$ and get

$$n > \frac{1 + 2s}{s^2 - 2}.$$

In total, we have just shown that

$$n > \frac{1 + 2s}{s^2 - 2} \quad \implies \quad \tilde{s}^2 > 2.$$

Therefore $\tilde{s} \notin A$, and by definition of A we have

$$\tilde{s} \geq q, \quad \forall q \in A.$$

Moreover $\tilde{s} := (s - 1/n) \in \mathbb{Q}$. Therefore $\tilde{s}$ is an upper bound of A in $\mathbb{Q}$. Since $s = \sup A$ is the smallest upper bound, see (3.16), it follows

$$s \leq \tilde{s}.$$

However

$$\tilde{s} := s - \frac{1}{n} < s,$$

obtaining a contradiction. Then $s \notin A$.

Conclusion.
We have assumed by contradiction that $s = \sup A$ exists in $\mathbb{Q}$. In this case either $s \in A$ or $s \notin A$. In both cases we found a contradiction. Therefore $\sup A$ does not exist.

The above theorem shows that the supremum does not necessarily exist. What about the infimum?

Question 3.37

Suppose $(K, +, \cdot, \leq)$ is an ordered field, and that $A \subseteq K$ is non-empty and bounded below. Does

$$\inf A$$

always exist?

The answer to the above question is again **NO**. A set bounded below for which the infimum does nor exist is, for example,

$$A = (\sqrt{2}, 10] \cap \mathbb{Q}.$$

The proof of this fact is, of course, very similar to the one of Theorem 3.36, and is therefore omitted.

Thus infimum and supremum do not exist in general. The fields for which all the bounded sets admit supremum or infimum are called **complete**.

Definition 3.38: Completeness

Let $(K, +, \cdot, \leq)$ be an ordered field. We say that K is **complete** if it holds the property:

- (AC) For every $A \subseteq K$ non-empty and bounded above

$$\sup A \in K.$$

Notation 3.39

We have that:

- Property (AC) is called **Axiom of Completeness**
- If K is an ordered field in which (AC) holds, then K is called a **complete ordered field**

Notice that if the **Axiom of Completeness** holds, then also the infimum exists. This is shown in the following proposition.

Proposition 3.40

Let $(K, +, \cdot, \leq)$ be a complete ordered field. Suppose that $A \subseteq K$ is non-empty and bounded below. Then

$$\inf A \in K.$$

Proof

Suppose that $A \subseteq K$ is non-empty and bounded below. Then

$$-A := \{-a : a \in A\}$$

is non-empty and bounded above. By completeness we have that $\sup(-A)$ exists in K. But then Proposition 3.34 implies that $\inf A$ exists in K, with

$$\inf A = -\sup(-A).$$

3.6 Equivalence of Completeness and Cut Property

We can show that **Completeness** is equivalent to the **Cut Property**. Such result is not essential, but its proof is very instructive.

Theorem 3.41: Equivalence of Cut Property and Completeness

Let $(K, +, \cdot, \leq)$ be an ordered field. Then they are equivalent:

1. K has the **Cut Property**
2. K is **Complete**

Remark 3.42: Ideas for proving Theorem 3.41

The proof of Theorem 3.41 is rather long, but the ideas are simple:
Step 1. Cut Property $\implies$ Completeness. Suppose K has the Cut Property. To prove that K is Complete, we need to:

- Consider an arbitrary set $A \subseteq K$ such that $A \neq \emptyset$ and A is bounded above.

- Show that A has a supremum.

To achieve this, consider the set

$$B := \{b \in K \,:\, b \geq a\,, \;\forall a \in A\}\,,$$

which is the set of Upper Bounds of A. We can show that the pair

$$(B^c, B)$$

is a Cut of K. As K has the Cut Property, then there exists $s \in K$ separator of (B^c, B). We will show that the separator s is the supremum of A

$$s = \sup A\,.$$

Thus K is complete. See Figure 3.7 for a schematic picture of the above construction.
Step 2. Completeness $\implies$ Cut Property. Conversely, suppose that K is Complete. To prove that K has the Cut Property, we need to:

- Consider a cut (A, B) of K.
- Show that (A, B) has a separator $s \in K$.

This implication is easier. Indeed, since A is non-empty and bounded above, by Completeness there exists

$$\sup A \in K\,.$$

We will show that

$$s := \sup A$$

is a separator for the cut (A, B). See Figure 3.8 for a schematic picture of the above construction.

Keeping the above ideas in mind, let us proceed with the proof.
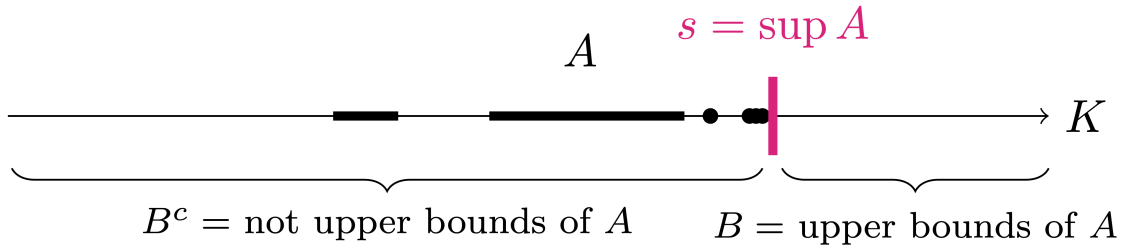
Figure 3.7: Let s be the separator of the cut (B^c, B), with B the set of upper bounds of A. Then $s = \sup A$.
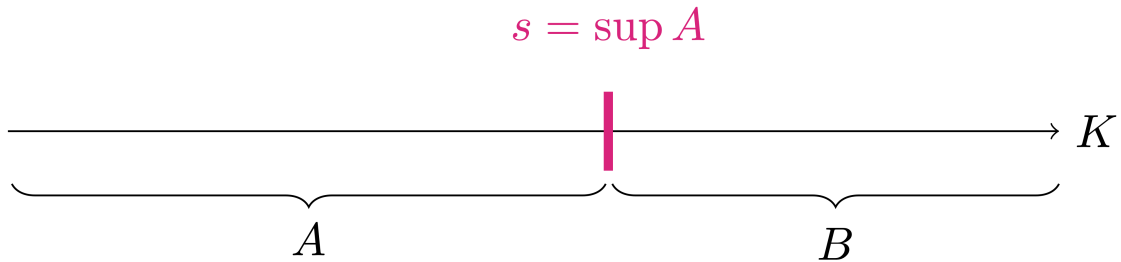
Figure 3.8: Let (A, B) be a cut of K and let $s = \sup A$. Then s is the separator of the cut (A, B).

Proof: Proof of Theorem 3.41

Step 1. Cut Property $\implies$ Completeness.
We need to prove that K is complete. To this end, consider $A \subseteq K$ non-empty and bounded above. Define the set of upper bounds of A:

$$B := \{b \in K \ : \ b \geq a, \ \forall a \in A\}.$$

Claim. The pair (B^c, B) is a cut of K.
Proof of Claim. We have to prove two points:

- (B^c, B) forms a partition of K.

  Indeed, we have $B \neq \emptyset$, since A is bounded above. Further, we have $B^c \neq \emptyset$, since A is non-empty. Thus

  $$K = B^c \cup B, \quad B^c \cap B = \emptyset.$$

  Then (B^c, B) is a partition of K.

- We have

  $$x \leq y, \quad \forall\, x \in B^c, \, \forall\, y \in B. \tag{3.19}$$

  To show the above, let $x \in B^c$ and $y \in B$. By definition of B we have that elements of B^c are not upper bounds of A. Therefore x is not an upper bound. This means there exists $\tilde{a} \in A$ which is larger than x, that is,

  $$x \leq \tilde{a}.$$

Since $y \in B$, then y is an upper bound for A, so that

$$a \leq y \,, \ \forall a \in A \,.$$

Therefore

$$x \leq \tilde{a} \leq y \,,$$

concluding (3.19).

Thus (B^c, B) is a cut of K and the claim is proven.
Since (B^c, B) is a cut of K, by the Cut Property there exists a separator $s \in K$ such that

$$x \leq s \leq y \,, \quad \forall \, x \in B^c \,, \ \forall \, y \in B \,. \tag{3.20}$$

Claim. s is an upper bound for A.
Proof of Claim.
Suppose by contradiction that s is not an upper bound for A. Therefore by definition of upper bound, there exists $\tilde{a} \in A$ such that

$$s < \tilde{a} \,.$$

Consider the mid-point between s and $\tilde{a}$, that is,

$$m := \frac{s + \tilde{a}}{2} \in K \,.$$

Since m is the mid-point between s and $\tilde{a}$, and $s < \tilde{a}$, it holds

$$s < m < \tilde{a} \,.$$

Indeed, since $s < \tilde{a}$ then

$$s = \frac{2s}{2} < \frac{s + \tilde{a}}{2} < \frac{2\tilde{a}}{2} = \tilde{a} \,.$$

In particular the above tells us that m is not an upper bound for A, given that $\tilde{a} \in A$ and $m < \tilde{a}$. Therefore $m \in B^c$, by definition of B^c. Therefore (3.20) implies

$$m \leq s \,,$$

which contradicts $s < m$. Hence s is an upper bound of A, concluding the proof of Claim.
Conclusion. We have shown that s is an upper bound of A. Condition (3.20) tells us that

$$s \leq y \,, \ \forall y \in B \,.$$

Recalling that B is the set of upper bounds of A, this means that s is the smallest upper bound of A, that is,

$$s = \sup A \in K \,.$$

Step 2. Completeness $\implies$ Cut Property.
Suppose K is complete. We need to show that K has the Cut Property. Therefore assume (A, B) is a cut of K, that is,

$$A \neq \emptyset, \quad B \neq \emptyset,$$

$$K = A \cup B, \quad A \cap B = \emptyset,$$

$$a \leq b, \quad \forall\, a \in A, \ \forall\, b \in B. \tag{3.21}$$

Since $B \neq \emptyset$, from (3.21) it follows that A is bounded above: indeed, every element of B is an upper bound for A, thanks to (3.21). Since $A \neq \emptyset$, by the Axiom of Completeness we have

$$s = \sup A \in K.$$

In particular, by definition of supremum, we have

$$a \leq s, \ \forall\, a \in A.$$

Let now $b \in B$ be arbitrary. From (3.21) we have that

$$a \leq b, \ \forall\, a \in A. \tag{3.22}$$

Therefore b is an upper bound of A. Since $s = \sup A$, we have that s is the smallest upper bound, and so

$$s \leq b.$$

Given that $b \in B$ was arbitrary, it actually holds

$$s \leq b, \ \forall\, b \in B. \tag{3.23}$$

From (3.22) and (3.23) we therefore have

$$a \leq s \leq b, \ \forall a \in A, \ \forall b \in B,$$

showing that s is a separator of (A, B). Thus K has the Cut Property.

3.7 Axioms of Real Numbers

We now have all the key elements to introduce the Real Numbers $\mathbb{R}$. These ingredients are:

- Definition of **ordered field**,
- The **Cut Property** or **Axiom of Completeness**.

The definition of $\mathbb{R}$ is given in an axiomatic way.

Definition 3.43:   System of Real Numbers $\mathbb{R}$

A system of Real Numbers is a set $\mathbb{R}$ satisfying the following properties:

1. There is an operation $+$ of **addition** on $\mathbb{R}$

$$+ \; : \; \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad (x, y) \mapsto x + y$$

   The addition satisifes: $\forall\, x, y, z \in \mathbb{R}$

   - (A1) **Commutativity and Associativity**:

$$x + y = y + x$$

$$(x + y) + z = x + (y + z)$$

   - (A2) **Additive Identity**: $\exists\, 0 \in \mathbb{R}$ s.t.

$$x + 0 = 0 + x = x$$

   - (A3) **Additive Inverse**: $\exists\, (-x) \in \mathbb{R}$ s.t.

$$x + (-x) = (-x) + x = 0$$

2. There is an operation $\cdot$ of **multiplication** on $\mathbb{R}$

$$\cdot \; : \; \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad (x, y) \mapsto x \cdot y = xy$$

   The multiplication satisifes: $\forall\, x, y, z \in \mathbb{R}$

   - (M1) **Commutativity and Associativity**:

$$x \cdot y = y \cdot x$$

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

   - (M2) **Multiplicative Identity**: $\exists\, 1 \in \mathbb{R}$ s.t.

$$x \cdot 1 = 1 \cdot x = x$$

   - (M3) **Multiplicative Inverse**: If $x \neq 0$, $\exists\, x^{-1} \in \mathbb{R}$ s.t.

$$x \cdot x^{-1} = x^{-1} \cdot x = 1$$

3. There is a relation $\leq$ of **total order** on $\mathbb{R}$. The order satisfies: $\forall\, x, y, z \in \mathbb{R}$

   - (O1) **Reflexivity**:
$$x \leq x$$

   - (O2) **Antisymmetry**:
$$x \leq y \text{ and } y \leq x \implies x = y$$

- (O3) **Transitivity**:

$$x \leq y \ \text{ and } \ y \leq z \ \implies \ x = z$$

- (O4) **Total order**:

$$x \leq y \ \text{ or } \ y \leq x$$

4. The operations $+$ and $\cdot$, and the total order $\leq$, are related by the following properties: $\forall x, y, z \in \mathbb{R}$

- (AM) **Distributive**: Relates addition and multiplication via

$$x \cdot (y + z) = x \cdot y + x \cdot z$$

- (AO) Relates addition and order with the requirement:

$$x \leq y \ \implies \ x + z \leq y + z$$

- (MO) Relates multiplication and order with the requirement:

$$x \geq 0, \ y \geq 0 \ \implies \ x \cdot y \geq 0$$

5. **Cut Property** holds:

- (CP) Every cut (A, B) of $\mathbb{R}$ admits a **separator** $s \in \mathbb{R}$ s.t.

$$a \leq s \leq b, \quad \forall \, a \in A, \, \forall \, b \in B$$

Remark 3.44

Since **Cut Property** and **Axiom of Completeness** are equivalent by Theorem 3.41, one can replace the Cut Property in Definition 3.43 Point 5 with:

5. **Axiom of Completeness** holds:

- (AC) For every $A \subseteq \mathbb{R}$ non-empty and bounded above

$$\sup A \in \mathbb{R}$$

Notation 3.45

For $x \in \mathbb{R}$, $x \neq 0$, the multiplicative inverse is also denoted by

$$x^{-1} = \frac{1}{x} \, .$$

> **Remark 3.46**
>
> Recall that
>
> - $(K, +, \cdot)$ satisfying
>
> $$\text{(A1)-(A3), (M1)-(M3), (AM)}$$
>
>   is a **field**
> - $(K, +, \cdot, \geq)$ satisfying
>
> $$\text{(A1)-(A3), (M1)-(M3), (O1)-(O4) ,(AM), (AO), (MO)}$$
>
>   is an **ordered field**
>
> In particular we have that
> $$(\mathbb{R}, +, \cdot, \leq)$$
> is a **complete ordered field**: that is, an ordered field in which the **Cut Property** (CP) or **Axiom of Completeness** (AC) hold

> **Important**
>
> It can be shown that $(\mathbb{R}, +, \cdot, \leq)$ is the **only** complete ordered field.
>
> > The above has to be intended in the following sense: if $(K, +, \cdot, \geq)$ is another complete ordered field, then *K looks like* $\mathbb{R}$. Mathematically this means that there exists an invertible map $\Psi : \mathbb{R} \to K$, called isomorphism of fields, which preserves the operations $+, \cdot$ and the order $\leq$.

> **Question 3.47**
>
> We have only postulated the existence of $\mathbb{R}$. Does such complete ordered field actually exist?

The answer is **YES**. There are several equivalent models for the system $\mathbb{R}$. If time allows, we will look into one of these models at the end of the module.

3.8 Special subsets of $\mathbb{R}$

In Definition 3.43 we have introduced $\mathbb{R}$ as a complete ordered field. This was done axiomatically and in a non constructive way. What happens now to the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$? Are they well defined? Does it still hold that

$$\mathbb{N}, \mathbb{Z}, \mathbb{Q} \subseteq \mathbb{R} ?$$

The definitions that we gave in Chapter 1 for $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ are not related to the system of real numbers $\mathbb{R}$ we just introduced. To overcome this problem, we will have to define new sets

$$\mathbb{N}_{\mathbb{R}}, \ \mathbb{Z}_{\mathbb{R}}, \ \mathbb{Q}_{\mathbb{R}}$$

from scracth, starting from the axioms of $\mathbb{R}$. Note that we are using the subscript $\mathbb{R}$ to distinguish these new sets from the old ones.

3.8.1 Natural numbers

Let us start with the definition of $\mathbb{N}_{\mathbb{R}}$. We would like $\mathbb{N}_{\mathbb{R}}$ to be

$$\mathbb{N}_{\mathbb{R}} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, ...\}.$$

Note that we are denoting the above numbers with bold symbols in order to distinguish them from the elements of $\mathbb{R}$. The key property that we would like $\mathbb{N}_{\mathbb{R}}$ to have is the following:

$$\text{Every } \mathbf{n} \in \mathbb{N}_{\mathbb{R}} \text{ has a successor } (\mathbf{n} + \mathbf{1}) \in \mathbb{N}_{\mathbb{R}}.$$

How do we ensure this property? We could start by defining

$$\mathbf{1} := 1,$$

with 1 the neutral element of the multiplication in $\mathbb{R}$, which exists by the field axiom (M2) in Defintion 3.43. We could then define $\mathbf{2}$ by setting

$$\mathbf{2} := 1 + 1.$$

We need a formal definition to capture this idea. This is the concept of **inductive set**.

Definition 3.48:   Inductive set

Let $S \subseteq \mathbb{R}$. We say that S is an inductive set if they are satisfied:

- $1 \in S$,
- If $x \in S$, then $(x + 1) \in S$.

Note that in the above definition we just used:

- The existence of the neutral element 1, given by axiom (M2).
- The operation of sum in $\mathbb{R}$, which is again given as an axiom.

Example 3.49

We have that

- $\mathbb{R}$ is an inductive set.

  Indeed we have $1 \in \mathbb{R}$ by axiom (M2). Moreover $(x + 1) \in \mathbb{R}$ for every $x \in \mathbb{R}$, by definition of

---

sum +.

- The set $A = \{0, 1\}$ is not an inductive set.

  This is because $1 \in A$, but $(1 + 1) \notin A$, since $1 + 1 \neq 0$.

Therefore $\mathbb{R}$ is an inductive set, showing that the definition of inductive set is not sufficient to fully describe our intuitive idea of $\mathbb{N}_{\mathbb{R}}$. The right way to define $\mathbb{N}_{\mathbb{R}}$ is as follows:

$$\mathbb{N}_{\mathbb{R}} \text{ is the smallest inductive subset of } \mathbb{R}.$$

To make the above definition formal we need a few observations.

Proposition 3.50

Let $\mathcal{M}$ be a collection of inductive subsets of $\mathbb{R}$. Then

$$S := \bigcap_{M \in \mathcal{M}} M$$

is an inductive subset of $\mathbb{R}$.

Proof

We have to show that the two properties of inductive sets hold for S:

- We have $1 \in M$ for every $M \in \mathcal{M}$, since these are inductive sets. Thus

$$1 \in \bigcap_{M \in \mathcal{M}} M = S.$$

- Suppose that $x \in S$. By definition of S this implies that $x \in M$ for all $M \in \mathcal{M}$. Since M is an inductive set, then $(x + 1) \in M$. Therefore $(x + 1) \in M$ for all $M \in \mathcal{M}$, showing that $(x + 1) \in S$.

Therefore S is an inductive set.

We are now ready to define the natural numbers $\mathbb{N}_{\mathbb{R}}$.

Definition 3.51: Set of Natural Numbers

Let $\mathcal{M}$ be the collection of **all** inductive subsets of $\mathbb{R}$. We define the set of natural numbers in $\mathbb{R}$ as

$$\mathbb{N}_{\mathbb{R}} := \bigcap_{M \in \mathcal{M}} M .$$

Therefore $\mathbb{N}_{\mathbb{R}}$ is the intersection of all the inductive subsets of $\mathbb{R}$. From this definition it follows that $\mathbb{N}_{\mathbb{R}}$ is the smallest inductive subset of $\mathbb{R}$, as shown in the following proposition.

Proposition 3.52: $\mathbb{N}_{\mathbb{R}}$ is the smallest inductive subset of $\mathbb{R}$

Let $C \subseteq \mathbb{R}$ be an inductive subset. Then
$$\mathbb{N}_{\mathbb{R}} \subseteq C .$$
In other words, $\mathbb{N}_{\mathbb{R}}$ is the smallest inductive set in $\mathbb{R}$.

Proof

Let $\mathcal{M}$ be the collection of **all** inductive subsets of $\mathbb{R}$. By definition

$$\mathbb{N}_{\mathbb{R}} = \bigcap_{M \in \mathcal{M}} M .$$

Let $x \in \mathbb{N}_{\mathbb{R}}$, then $x \in M$ for all $M \in \mathcal{M}$. Since $C \in \mathcal{M}$ then $x \in C$. This shows $\mathbb{N}_{\mathbb{R}} \subseteq C$.

The definition of $\mathbb{N}_{\mathbb{R}}$ guarantees that all numbers in $\mathbb{N}_{\mathbb{R}}$ are larger than 1.

Theorem 3.53

Let $x \in \mathbb{N}_{\mathbb{R}}$. Then
$$x \geq 1 .$$

Proof

Define the set
$$C := \{x \in \mathbb{R} : x \geq 1\} .$$
We have that C is an inductive subset of $\mathbb{R}$.

By definition $1 \in C$. Suppose now that $x \in C$, so that $x \geq 1$. Since $1 \geq 0$ as a consequence of

the field axioms, we deduce that

$$x + 1 \geq x + 0 = x \geq 1 \,,$$

showing that $x + 1 \geq 1$. Thus $(x + 1) \in C$.

By Proposition 3.52 we conclude that
$$\mathbb{N}_\mathbb{R} \subseteq C \,,$$

showing that $x \geq 1$ for all $x \in \mathbb{N}_\mathbb{R}$.

Notation 3.54

We have just shown that all the numebers $x \in \mathbb{N}_\mathbb{R}$ satisfy

$$x \geq 1 \,.$$

Moreover by the fact that $\mathbb{N}_\mathbb{R}$ is an inductive set, we know that

$$1 + 1 \in \mathbb{N}_\mathbb{R} \,,$$

since $1 \in \mathbb{N}_\mathbb{R}$. We denote
$$2 := 1 + 1 \,.$$

Similarly, we will have that
$$2 + 1 \in \mathbb{N}_\mathbb{R} \,,$$

since $2 \in \mathbb{N}_\mathbb{R}$. We denote
$$3 := 2 + 1 \,.$$

In this way we give a name to all the numbers in $\mathbb{N}_\mathbb{R}$.

3.8.2 Principle of induction

The Principle of Induction is a consequence of the definition of $\mathbb{N}_\mathbb{R}$, see Definition 3.51, and of the field axioms of $\mathbb{R}$ in Definition 3.43.

Theorem 3.55: Principle of Induction

Let $\alpha(n)$ be a statement depending on $n \in \mathbb{N}_\mathbb{R}$. Assume that

1. $\alpha(1)$ is true.

---

2. If $\alpha(n)$ is true then also $\alpha(n + 1)$ is true.

Then $\alpha(n)$ is true for all $n \in \mathbb{N}_{\mathbb{R}}$.

Proof

Define the set
$$C := \{x \in \mathbb{N}_{\mathbb{R}} \; : \; \alpha(n) \text{ is true}\}.$$

We have that C is an inductive subset of $\mathbb{R}$.

Indeed:

- $1 \in C$ since $\alpha(1)$ is true by assumption.
- If $n \in C$ then $\alpha(n)$ is true. By assumption $\alpha(n + 1)$ is true. Therefore $(n + 1) \in C$.

By Proposition 3.52 we conclude that
$$\mathbb{N}_{\mathbb{R}} \subseteq C.$$

As by definition $C \subseteq \mathbb{N}_{\mathbb{R}}$, we have proven that

$$\mathbb{N}_{\mathbb{R}} = C,$$

showing that $\alpha(n)$ is true for all $n \in \mathbb{N}_{\mathbb{R}}$.

As a consequence of the principle of induction, we can prove that $\mathbb{N}_{\mathbb{R}}$ is closed under the field operations of sum and multiplication.

Theorem 3.56

For all $n, m \in \mathbb{N}_{\mathbb{R}}$ we have:

1. $\mathbb{N}_{\mathbb{R}}$ is closed under addition, that is,
$$m + n \in \mathbb{N}_{\mathbb{R}}.$$

2. $\mathbb{N}_{\mathbb{R}}$ is closed under multiplication, that is,
$$m \cdot n \in \mathbb{N}_{\mathbb{R}},$$

3. If $m > n$ there exists $k \in \mathbb{N}_{\mathbb{R}}$ such that
$$m = n + k.$$

Proof

We only prove the first point, the other statements are left as an exercise. Fix $m \in \mathbb{N}_\mathbb{R}$. We prove that

$$m + n \in \mathbb{N}_\mathbb{R}, \quad \forall n \in \mathbb{N}_\mathbb{R}, \tag{3.24}$$

by using induction.

- Induction base: We have $m + 1 \in \mathbb{N}_\mathbb{R}$, since $m \in \mathbb{N}_\mathbb{R}$ and $\mathbb{N}_\mathbb{R}$ is an inductive set.
- Inductive hypothesis: Suppose $m + n \in \mathbb{N}_\mathbb{R}$. Since $\mathbb{N}_\mathbb{R}$ is an inductive set, we have $(m + n) + 1 \in \mathbb{N}_\mathbb{R}$. By associativity of the sum, see axiom (A1), we get

$$m + (n + 1) = (m + n) + 1 \in \mathbb{N}_\mathbb{R},$$

which is the desired theis.

By the Induction Principle of Theorem 3.55 we conclude (3.24).

As a consequence of the above theorem, we see that the restriction of the operations of sum and multiplication to $\mathbb{N}_\mathbb{R}$ are still binary operations:

$$+ \, : \, \mathbb{N}_\mathbb{R} \times \mathbb{N}_\mathbb{R} \to \mathbb{N}_\mathbb{R}, \quad \cdot \, : \, \mathbb{N}_\mathbb{R} \times \mathbb{N}_\mathbb{R} \to \mathbb{N}_\mathbb{R}.$$

Equipped with the above operations, $\mathbb{N}_\mathbb{R}$ satisfies the following properties.

Theorem 3.57

$(\mathbb{N}_\mathbb{R}, +, \cdot, \leq)$ satisfies the following axioms from Definition 3.43:

- (A1).
- (M1), (M2).
- (O1)-(O4).
- (AM), (AO), (MO).

The proof is trivial, as it follows immediately from the inclusion of $\mathbb{N}_\mathbb{R}$ in $\mathbb{R}$.

3.8.3 Integers

We have seen in Theorem 3.56 that $\mathbb{N}_\mathbb{R}$ is closed under addition. However $\mathbb{N}_\mathbb{R}$ is not closed under subtraction. We therefore define the set of **integers** $\mathbb{Z}_\mathbb{R}$ in a way that we can perform subtraction of any two natural numbers.

Definition 3.58: Set of Integers

The set of integers in $\mathbb{R}$ is defined by

$$\mathbb{Z}_{\mathbb{R}} := \{m - n : n, m \in \mathbb{N}_{\mathbb{R}}\}.$$

In the definition of $\mathbb{Z}_{\mathbb{R}}$ we denote by $-n$ the inverse of n in $\mathbb{R}$, which exists by the field axiom (A3) in Definition 3.43. The following characterization explains the relationship between $\mathbb{Z}_{\mathbb{R}}$ and $\mathbb{N}_{\mathbb{R}}$.

Theorem 3.59

It holds

$$\mathbb{Z}_{\mathbb{R}} = \{-n : n \in \mathbb{N}_{\mathbb{R}}\} \cup \{0\} \cup \mathbb{N}_{\mathbb{R}}.$$

Proof

Define the set

$$M := \{-n : n \in \mathbb{N}_{\mathbb{R}}\} \cup \{0\} \cup \mathbb{N}_{\mathbb{R}}.$$

- $M \subseteq \mathbb{Z}_{\mathbb{R}}$: Suppose $m \in M$. We have 3 cases:
    - If $m \in \{-n : n \in \mathbb{N}_{\mathbb{R}}\}$ then there exists $n \in \mathbb{N}_{\mathbb{R}}$ such that $m = -n$. Thus
    $$m = -n = 1 - (n + 1) \in \mathbb{Z}_{\mathbb{R}},$$
    since $1 \in \mathbb{N}_{\mathbb{R}}$ and $n + 1 \in \mathbb{N}_{\mathbb{R}}$ because $n \in \mathbb{N}_{\mathbb{R}}$.
    - If $m = 0$ then
    $$m = 0 = 1 - 1 \in \mathbb{Z}_{\mathbb{R}},$$
    as $1 \in \mathbb{N}_{\mathbb{R}}$.
    - If $m \in \mathbb{N}_{\mathbb{R}}$ then
    $$m = (m + 1) - 1 \in \mathbb{Z}_{\mathbb{R}},$$
    since $1 \in \mathbb{N}_{\mathbb{R}}$ and $m + 1 \in \mathbb{N}_{\mathbb{R}}$, given that $m \in \mathbb{N}_{\mathbb{R}}$.

    In all 3 cases we have shown that $m \in \mathbb{Z}_{\mathbb{R}}$, proving that $M \subseteq \mathbb{Z}_{\mathbb{R}}$.

- $\mathbb{Z}_{\mathbb{R}} \subseteq M$: Let $z \in \mathbb{Z}_{\mathbb{R}}$. Then $z = m - n$ for some $n, m \in \mathbb{N}_{\mathbb{R}}$. We have 3 cases:
    - If $m = n$ then
    $$z = m - n = m - m \stackrel{(A3)}{=} 0 \in M.$$
    - If $m > n$, by Theorem 3.56 there exists $k \in \mathbb{N}_{\mathbb{R}}$ such that $m = k + n$. Therefore
    $$z = m - n = (k + n) - n$$
    $$\stackrel{(A1)}{=} k + (n - n) \stackrel{(A3)}{=} k + 0$$
    $$\stackrel{(A2)}{=} k \in M,$$

since $k \in \mathbb{N}_\mathbb{R}$.

– If $m < n$, by Theorem 3.56 there exists $k \in \mathbb{N}_\mathbb{R}$ such that $n = k + m$. Therefore

$$z = m - n = -k \in M \,,$$

since $k \in \mathbb{N}_\mathbb{R}$, where again we have used (implicitly) the field axioms (A1), (A2) and (A3).

Therefore $\mathbb{Z}_\mathbb{R} = M$.

Like we did with $\mathbb{N}_\mathbb{R}$, we can also show that $\mathbb{Z}_\mathbb{R}$ is closed under the operations of sum and multiplication.

Theorem 3.60

For all $n, m \in \mathbb{Z}_\mathbb{R}$ we have:

1. $\mathbb{Z}_\mathbb{R}$ is closed under addition, that is,
$$m + n \in \mathbb{Z}_\mathbb{R} \,.$$

2. $\mathbb{Z}_\mathbb{R}$ is closed under multiplication, that is,
$$m \cdot n \in \mathbb{Z}_\mathbb{R} \,,$$

The proof is left as an exercise. As a consequence of Theorem 3.60 we have that the restriction of the operations of sum and multiplication to $\mathbb{Z}_\mathbb{R}$ are still binary operations:

$$+ \,:\, \mathbb{Z}_\mathbb{R} \times \mathbb{Z}_\mathbb{R} \to \mathbb{Z}_\mathbb{R} \,, \quad \cdot \,:\, \mathbb{Z}_\mathbb{R} \times \mathbb{Z}_\mathbb{R} \to \mathbb{Z}_\mathbb{R} \,.$$

Equipped with the above operations, $\mathbb{Z}_\mathbb{R}$ satisfies the following properties.

Theorem 3.61

$(\mathbb{Z}_\mathbb{R}, +, \cdot, \leq)$ satisfies the following axioms from Definition 3.43:

- (A1), (A2), (A3).
- (M1), (M2).
- (O1)-(O4).
- (AM), (AO), (MO).

Proof

The fact that

$$\text{(A1), (A2), (M1), (M2), (O1)-(O4), (AM), (AO), (MO)}$$

are satisfied descends immediately from the inclusion

$$\mathbb{Z}_{\mathbb{R}} \subseteq \mathbb{R}\,.$$

We are left to prove (A3). This is non-trivial because a priori the additive inverse $-z$ of some $z \in \mathbb{Z}_{\mathbb{R}}$ belongs to $\mathbb{R}$. We need to check that $-z \in \mathbb{Z}_{\mathbb{R}}$. Indeed, since $z \in \mathbb{Z}_{\mathbb{R}}$, there exist $n, m \in \mathbb{N}_{\mathbb{R}}$ such that $z = m - n$. Define $y := n - m$. We have that $y \in \mathbb{Z}_{\mathbb{R}}$ and

$$z + y = (m - n) + (n - m) = (m - m) + (n - n) = 0\,.$$

Therefore y is the inverse of z and $y \in \mathbb{Z}_{\mathbb{R}}$, proving that the sum in $\mathbb{Z}_{\mathbb{R}}$ satisfies (A3).

Remark 3.62

$\mathbb{Z}_{\mathbb{R}}$ does not satisfy (M3).

For example, let us show that $2 \in \mathbb{Z}_{\mathbb{R}}$ has no inverse in $\mathbb{Z}_{\mathbb{R}}$. Indeed, let $m \in \mathbb{Z}_{\mathbb{R}}$. By Theorem 3.59 we have 3 cases:

- $m \in \mathbb{N}_{\mathbb{R}}$: Since $2 > 1$ we have
$$2 \cdot m > 1 \cdot m \geq 1$$
where in the last inequality we used that $m \geq 1$ for all $m \in \mathbb{N}_{\mathbb{R}}$, as shown in Theorem 3.53. The above shows that
$$2 \cdot m > 1\,,$$
and therefore m cannot be the inverse of 2.

- $m = 0$: Then $2 \cdot m = 0$, so that m cannot be the inverse of 2.

- $m = -n$ with $n \in \mathbb{N}_{\mathbb{R}}$. Then
$$2 \cdot m = 2 \cdot (-n) < 0\,,$$
so that m cannot be the inverse of 2.

As we have exhausted all the possibilities, we conclude that 2 does not have a multiplicative inverse in $\mathbb{N}_{\mathbb{R}}$.

3.8.4 Rational numbers

In Theorem 3.61 and 3.62 we have seen that $\mathbb{Z}_{\mathbb{R}}$ satisfy all the field axiom, except for (M3). We therefore extend $\mathbb{Z}_{\mathbb{R}}$ in a way that the extension contains multiplicative inverses. The extension is the set of rational numbers $\mathbb{Q}_{\mathbb{R}}$.

> **Definition 3.63:**   Set of Rational Numbers
>
> The set of rational numbers in $\mathbb{R}$ is
>
> $$\mathbb{Q}_{\mathbb{R}} := \left\{ \frac{m}{n} \ : \ m \in \mathbb{Z}_{\mathbb{R}}, \ n \in \mathbb{N}_{\mathbb{R}} \right\}.$$

Notice that in the above definition we are just using the field axiom (M3), with

$$\frac{m}{n} := m \cdot n^{-1}.$$

The inverse of n exists because we are assuming $n \in \mathbb{N}_{\mathbb{R}}$, and therefore n cannot be 0, as a consequence of Theorem 3.53.

The set $\mathbb{Q}_{\mathbb{R}}$ is closed under addition and multiplication (exercise). Therefore they are well defined the operations:

$$+ \ : \ \mathbb{Q}_{\mathbb{R}} \times \mathbb{Q}_{\mathbb{R}} \to \mathbb{Q}_{\mathbb{R}}, \quad \cdot \ : \ \mathbb{Q}_{\mathbb{R}} \times \mathbb{Q}_{\mathbb{R}} \to \mathbb{Q}_{\mathbb{R}}.$$

> **Theorem 3.64**
>
> $(\mathbb{Q}_{\mathbb{R}}, +, \cdot, \leq)$ is an ordered field.

> **Proof**
>
> All the field properties, except for (M3), follow from the inclusion
>
> $$\mathbb{Q}_{\mathbb{R}} \subseteq \mathbb{R}$$
>
> and from the field properties of $\mathbb{R}$. To check (M3), let $q \in \mathbb{Q}_{\mathbb{R}}$ with $q \neq 0$. Therefore $q = m/n$ for $m \in \mathbb{Z}_{\mathbb{R}}$, $n \in \mathbb{N}_{\mathbb{R}}$. As $q \neq 0$ and $n \neq 0$, see Theorem 3.53, we deduce that $m \neq 0$. We have two cases:
>
> - $m > 0$: In this case $m \in \mathbb{N}_{\mathbb{R}}$ by Theorem 3.59. Therefore
>
> $$p = \frac{n}{m} \in \mathbb{Q}_{\mathbb{R}}$$
>
>   by definition, since $n, m \in \mathbb{N}_{\mathbb{R}}$. By commutativity we have
>
> $$q \cdot p = \frac{m}{n} \cdot \frac{n}{m} = 1.$$
>
> - $m < 0$: Then $m = -x$ with $x \in \mathbb{N}_{\mathbb{R}}$ by Theorem 3.59. Therefore
>
> $$p = \frac{-n}{x} \in \mathbb{Q}_{\mathbb{R}}$$
>
>   by definition, since $-n \in \mathbb{Z}_{\mathbb{R}}$ and $x \in \mathbb{N}_{\mathbb{R}}$. By commutativity we have
>
> $$q \cdot p = \frac{m}{n} \cdot \frac{-n}{x} = \frac{m}{n} \cdot \frac{-n}{-m} = 1.$$
>
> Therefore q always admits a multiplicative inverse q^{-1} belonging to $\mathbb{Q}_{\mathbb{R}}$, proving (M3).

The set $\mathbb{Q}_\mathbb{R}$ does not have the Cut Property or the Axiom of Completeness.

Theorem 3.65

$\mathbb{Q}_\mathbb{R}$ is not complete.

The proof of the above Theorem is a one to one copy of the proof of Theorem 3.36: indeed the proof of Theorem 3.36 only makes use of field axioms, and thus it applies to $\mathbb{Q}_\mathbb{R}$.

Notation 3.66

From now on we denote
$$\mathbb{N} := \mathbb{N}_\mathbb{R}, \quad \mathbb{Z} := \mathbb{Z}_\mathbb{R}, \quad \mathbb{Q} := \mathbb{Q}_\mathbb{R},$$
dropping the subscript $\mathbb{R}$.

4 Properties of $\mathbb{R}$

Now that we established the axiomatic definition of the Real Numbers $\mathbb{R}$ as a **complete ordered field**, let us investigate some of the properties of $\mathbb{R}$. These will be consequence of the axioms of the real numbers, particulaly of the **Axiom of Completeness**.

4.1 Archimedean Property

The Archimedean property is one of the most useful properties of $\mathbb{R}$, and it essentially states that the set of natural numbers $\mathbb{N}$ is not bounded above in $\mathbb{R}$.

More precisley, the Archimedean Property says two things:

1. For any $x \in \mathbb{R}$ we can always find a natural number $n \in \mathbb{N}$ such that

$$n > x.$$

2. For any $x \in \mathbb{R}$ with $x > 0$, we can always find a natural number $m \in \mathbb{N}$ such that

$$0 < \frac{1}{m} < x.$$

The situation is depicted in Figure 4.2.

Figure 4.1: For any $x > 0$ we can find $n, m \in \mathbb{N}$ such that $1/m < x < n$.

> **Remark 4.1**
>
> The Archimedean property might sound trivial. However there are examples of ordered fields K that satisfy:
>
> 1. $\mathbb{N} \subseteq K$.
> 2. K does not have the Archimedean property.
> 3. In particular, $\mathbb{N}$ is bounded above in K.
>
> Of course such fields K cannot be complete.

If K is complete, then K is essentially $\mathbb{R}$, and we are going to prove the Archimedean Property holds in $\mathbb{R}$.

Let us proceed with the precise statement of the Archimedean property in $\mathbb{R}$.

Theorem 4.2:   Archimedean Property

Let $x \in \mathbb{R}$ be given. Then:

1. There exists $n \in \mathbb{N}$ such that
$$n > x .$$

2. Suppose in addition that $x > 0$. There exists $n \in \mathbb{N}$ such that
$$\frac{1}{n} < x .$$

Proof

Part 1. Let $x \in \mathbb{R}$. Suppose by contradiction that there is no $n \in \mathbb{N}$ such that
$$n > x .$$

This means that
$$n \leq x \quad \forall n \in \mathbb{N} . \tag{4.1}$$

The above is saying that the set $\mathbb{N}$ is bounded above. Since $\mathbb{N}$ is not empty, by the Axiom of Completeness there exists
$$\alpha = \sup \mathbb{N} .$$

Claim: $(\alpha - 1)$ is not an upper bound for $\mathbb{N}$.
Proof of Claim. Indeed, we have
$$(\alpha - 1) < \alpha . \tag{4.2}$$

Therefore $\alpha - 1$ cannot be an upper bound for $\mathbb{N}$. Indeed, if by contradiction $\alpha - 1$ was an upper bound for $\mathbb{N}$, then we would have
$$\alpha \leq (\alpha - 1) ,$$

since α is the smallest upper bound for $\mathbb{N}$. This contradicts (4.2). Therefore $\alpha - 1$ is not an upper bound for $\mathbb{N}$.
Conclusion. Since $\alpha - 1$ is not an upper bound for $\mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that
$$\alpha - 1 < n_0 .$$

The above implies
$$\alpha < n_0 + 1 .$$

Since
$$(n_0 + 1) \in \mathbb{N},$$
we have obtained a contradiction, given that α was the supremum of $\mathbb{N}$. Thus (4.1) is false, meaning that there exists $n \in \mathbb{N}$ such that
$$n > x.$$

Part 2. Suppose $x \in \mathbb{R}$ with $x > 0$. We can define

$$y := \frac{1}{x}.$$

By Part 1 there exists $n \in \mathbb{N}$ such that
$$n > y = \frac{1}{x}.$$
Using that $x > 0$, we can rearrange the above inequality to obtain

$$\frac{1}{n} < x,$$

which is the desired thesis.

There is another formulation of the Archimedean Property which, depending on the situation, might be more useful. This formulation says the following: If $x, y \in \mathbb{R}$ are such that
$$0 < x < y,$$
then there exists $n \in \mathbb{N}$ such that
$$nx > y.$$
In other words, if one does n steps of size x in the positive numbers direction, then the resulting number nx will be larger than y. The situation is depicted in Figure 4.2.
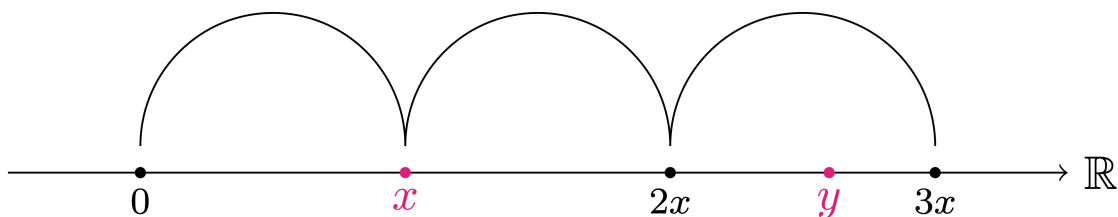
Figure 4.2: For $0 < x < y$ there exists $n \in \mathbb{N}$ such that that $nx > y$. In the picture $n = 3$.

Theorem 4.3: Archimedean Property (Alternative formulation)

Let $x, y \in \mathbb{R}$, with $0 < x < y$. There exists $n \in \mathbb{N}$ such that

$$nx > y.$$

Proof

Suppose by contradiction that there does not exist some $n \in \mathbb{N}$ such that

$$nx > y.$$

This means that

$$nx \leq y, \quad \forall n \in \mathbb{N}. \tag{4.3}$$

Define the set

$$A := \{nx : n \in \mathbb{N}\}.$$

Condition (4.3) is saying that A is bounded above by y. Morever A is trivially non-empty. By the Axiom of Completeness there exists

$$\alpha = \sup A.$$

Since α is the supremum of A, by definition of supremum and of the set A, we have

$$nx \leq \alpha, \quad \forall n \in \mathbb{N}. \tag{4.4}$$

As (4.4) holds for every $n \in \mathbb{N}$, then it also holds for $(n+1)$, meaning that

$$(n+1)x \leq \alpha.$$

The above implies

$$nx \leq \alpha - x.$$

As n was arbitrary, we conclude that

$$nx \leq \alpha - x, \quad \forall n \in \mathbb{N}.$$

The above is saying that $(\alpha - x)$ is an upper bound for A. Since α is the supremum of A, in particular α is the smallest upper bound. Thus it must hold

$$\alpha \leq \alpha - x.$$

The above is equivalent to

$$x \leq 0,$$

which contradicts our assumption of $x > 0$. Therefore (4.3) is false, and there exists $n \in \mathbb{N}$ such that

$$nx > y,$$

concluding the proof.

4.2 Nested Interval Property

Another consequence of the axiom of completeness is the **Nested Interval Property**. This is yet another way of saying the same thing: $\mathbb{R}$ does not have **gaps**.

Let us look at a construction. Suppose given some closed intervals

$$I_n := [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\},$$

where the end points are ordered in the following way:

$$a_1 \leq a_2 \leq ... \leq a_n \leq \cdots \leq b_n \leq ... b_n \leq b_1,$$

as shown in Figure 4.3.
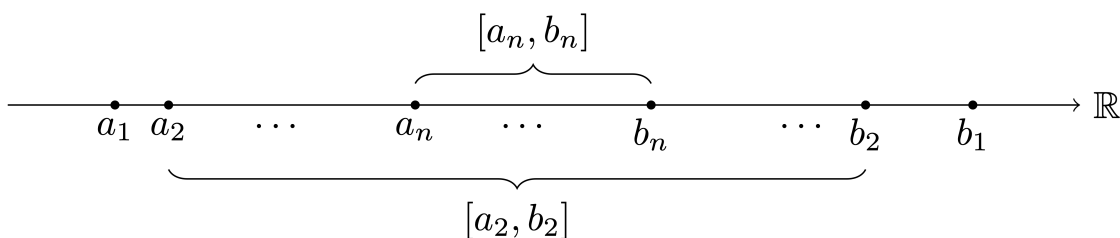
Figure 4.3: Nested intervals $I_n = [a_n, b_n]$.

The intervals I_n are **nested**, meaning that

$$I_1 \supset I_2 \supset I_3 \supset ... I_n \supset ...$$

For finite intersections we clearly have

$$\bigcap_{n=1}^{k} I_n = I_k,$$

that is, intersecting the first k intervals yields I_k, the smallest interval in the sequence.

Question 4.4

Consider the infinite intersection

$$\bigcap_{n=1}^{\infty} I_n := \{x \in \mathbb{R} : x \in I_n, \forall n \in \mathbb{N}\}.$$

What can we say about it? Is it empty? Is it not empty?

The answer is that the infinite intersection is not empty, because $\mathbb{R}$ was constructed in a way that it does not have gaps.

Theorem 4.5:   Nested Interval Property

For each $n \in \mathbb{N}$ assume given a closed interval

$$I_n := [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}.$$

Suppose that the intervals are nested, that is,

$$I_n \supset I_{n+1}, \quad \forall\, n \in \mathbb{N}.$$

Then

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset. \tag{4.5}$$

Proof

By definition we have

$$\bigcap_{n=1}^{\infty} I_n := \{x \in \mathbb{R} : x \in I_n, \forall\, n \in \mathbb{N}\}.$$

We want to prove (4.5). This means we need to find a real number x such that

$$x \in I_n, \quad \forall\, n \in \mathbb{N}. \tag{4.6}$$

Idea of the Proof: Condition (4.6) is saying that it should hold

$$x \geq a_n, \quad \forall\, n \in \mathbb{N}.$$

We might be tempted to choose x to be any of the b_n. This choice would indeed satisy the above. However (4.6) also implies that

$$x \leq b_n, \quad \forall\, n \in \mathbb{N}.$$

Therefore x has to be larger than all the a_n, but not too large. This suggests that x should be defined as a supremum.

Define the set

$$A := \{a_n : n \in \mathbb{N}\}.$$

The set A is non-empty and is bounded above by any of the b_n. Therefore there exists

$$x = \sup\, A.$$

By definition of supremum and definition of the set A, we have

$$a_n \leq x, \quad \forall\, n \in \mathbb{N}.$$

On the other hand, consider an arbitrary number b_n. By construction we have

$$a_i \leq b_n, \quad \forall i \in \mathbb{N}.$$

Therefore b_n is an upper bound for A. Since the supremum is the smallest upper bound, we conclude that

$$x \leq b_n.$$

The index n was chosen arbitrarily, and therefore

$$x \leq b_n, \quad \forall n \in \mathbb{N}.$$

In total we have

$$a_n \leq x \leq b_n, \quad \forall n \in \mathbb{N},$$

showing that x satisfies (4.6). Therefore (4.5) holds and the proof is concluded.

Important

The assumption that I_n is **closed** is crucial in Theorem 4.14. Without such assumption the thesis of Theorem 4.14 does not hold in general, as seen in Example 4.6 below.

Example 4.6

Consider the **open** intervals

$$I_n := \left(0, \frac{1}{n}\right).$$

These are clearly nested

$$I_n \supset I_{n+1}, \quad \forall n \in \mathbb{N}.$$

For this choice of I_n we have

$$\bigcap_{n=1}^{\infty} I_n = \varnothing. \tag{4.7}$$

Indeed, suppose by contradiction that the intersection is non-empty. Then there exists $x \in \mathbb{N}$ such that

$$x \in I_n, \quad \forall n \in \mathbb{N}.$$

By definition of I_n the above reads

$$0 < x < \frac{1}{n}, \quad \forall n \in \mathbb{N}. \tag{4.8}$$

Since $x > 0$, by the Archimedean Property in Theorem 4.11 Point 2, there exists $n_0 \in \mathbb{N}$ such that

$$0 < \frac{1}{n_0} < x.$$

The above contradicts (4.8). Therefore (4.7) holds.

4.3 Revisiting Sup and inf

We now investigate some of the properties of supremum and infimum in $\mathbb{R}$. The first property is an alternative characterization of the supremum, which we will often use. A sketch of such characterization is in Figure 4.4 below.

Proposition 4.7: Characterization of Supremum

Let $A \subseteq \mathbb{R}$ be a non-empty set. Suppose that $s \in \mathbb{R}$ is an upper bound for A. They are equivalent:

1. $s = \sup A$
2. For every $\varepsilon > 0$ there exists $x \in A$ such that

$$s - \varepsilon < x.$$
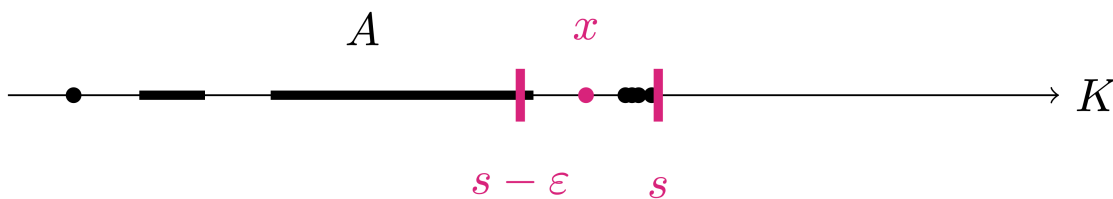
Figure 4.4: Let $s = \sup A$. Then for every $\varepsilon > 0$ there exist $x \in A$ such that $s - \varepsilon < x$.

Proof: Proof of Proposition 4.16

Step 1. Assume that
$$s = \sup A.$$
Let $\varepsilon > 0$ be arbitrary. We clearly have that
$$s - \varepsilon < s. \tag{4.9}$$
Therefore $(s - \varepsilon)$ cannot be an upper bound of A. Indeed, if by contradiction $(s - \varepsilon)$ was an upper bound, then we would have
$$s \leq (s - \varepsilon),$$

---

since s is the smallest upper bound. The above contradicts (4.9), and therefore $(s - \varepsilon)$ is not an upper bound for A. Hence there exists some $x \in A$ such that

$$s - \varepsilon < x,$$

concluding.

Step 2. Assume that Point 2 in the statement of Proposition 4.16 holds. By assumption we have that s is an upper bound for A. Suppose by contradiction that

$$s \neq \sup A.$$

This is equivalent to the statement

$$s \text{ is not the smallest upper bound of } A. \qquad (4.10)$$

Hence there exists an upper bound b of A such that

$$b < s.$$

Let

$$\varepsilon := s - b.$$

By assumption there exists $x \in A$ such that

$$s - \varepsilon < x.$$

Substituting the definition of ε we get

$$s - s + b < x \quad \implies \quad b < x.$$

Since b is an upper bound for A and $x \in A$, the above is a contradiction. Therefore (4.10) is false, and s is the smallest upper bound of A. Thus $s = \sup A$.

The analogue of Proposition 4.16 is as follows. The proof is left as an exercise.

Proposition 4.8: Characterization of Infimum

Let $A \subseteq \mathbb{R}$ be a non-empty set. Suppose that $i \in \mathbb{R}$ is a lower bound for A. They are equivalent:

1. $i = \inf A$
2. For every $\varepsilon \in \mathbb{R}$, with $\varepsilon > 0$, there exists $x \in A$ such that

$$x < i + \varepsilon.$$

A sketch of the characterization in Proposition 4.17 can be found in Figure 4.5 below.

With the above characterizations of supremum and infimum, it is now easier to prove that some candidate
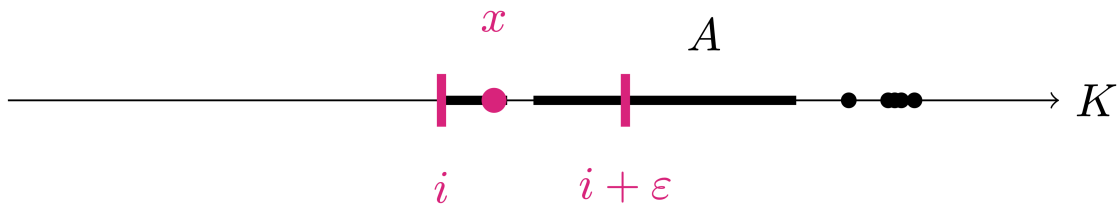
Figure 4.5: Let $i = \inf A$. Then for every $\varepsilon > 0$ there exist $x \in A$ such that $x < i + \varepsilon$.

number is the supremum or infimum of some set. As an example, let us characterize supremum and infimum of an open interval of $\mathbb{R}$.

Proposition 4.9

Let $a, b \in \mathbb{R}$ with $a < b$. Let
$$A := (a, b) = \{x \in \mathbb{R} : a < x < b\}.$$
Then
$$\inf A = a, \quad \sup A = b.$$

Proof

We will only prove that
$$\inf A = a,$$
since the proof of
$$\sup A = b$$
is similar.
By definition of A, we have that
$$a < x, \quad \forall x \in A.$$
The above says that a is a lower bound for A.
Claim. a is the largest lower bound of A.
Proof of Claim. Let L be a lower bound for A, that is,
$$L \leq x, \quad \forall x \in A.$$
We have to prove that
$$L \leq a. \tag{4.11}$$
Indeed suppose by contradiction that (4.11) does not hold, namely that
$$a < L.$$

Since $a < b$ we have

$$a = \frac{2a}{2} < \frac{a+b}{2} < \frac{2b}{b} = b,$$

showing that the midpoint between a and b satisfies

$$\frac{a+b}{2} \in A.$$

Since L is a lower bound for A we have

$$L \le \frac{a+b}{2} < b. \tag{4.12}$$

Consider the midpoint

$$M := \frac{a+L}{2}.$$

We have that

$$M \in A.$$

Indeed, recalling that $a < L$, we have

$$a = \frac{2a}{2} < \frac{a+L}{2} = M.$$

Moreover by (4.12) we have $L < b$. Thus

$$M = \frac{a+L}{2} \le \frac{a+b}{2} < b.$$

This shows $M \in A$.

Moreover

$$M < L.$$

This is because $a < L$, and therefore

$$M = \frac{a+L}{2} < \frac{2L}{2} = L.$$

This is a contradiction, since by assumption L is a lower bound for A, and thus we should have

$$L \le M.$$

Therefore (4.11) holds, showing that a is the largest lower bound of A. Thus $a = \inf A$.

As a corollary of the above we have that the maximum and minimum of an open interval do not exist.

Corollary 4.10

Let $a, b \in \mathbb{R}$ with $a < b$. Let
$$A := (a, b) = \{x \in \mathbb{R} : a < x < b\}.$$
Then $\min A$ and $\max A$ do not exist.

Proof

Suppose by contradiction that $\min A$ exists. We have shown that if the minimum of a set exists, then it must be
$$\min A = \inf A.$$
Since
$$\inf A = a,$$
by Proposition 4.9, we would obtain that
$$\min A = a.$$
By definition $\min A \in A$, so that $a \in A$. This is contradiction. Then $\min A$ does not exist. The proof that $\max A$ does not exist is similar, and is left as an exercise.

We can also consider intervals for which one or both of the sides are closed.

Corollary 4.11

Let $a, b \in \mathbb{R}$ with $a < b$. Let
$$A := [a, b) = \{x \in \mathbb{R} : a \leq x < b\}.$$
Then
$$\min A = \inf A = a, \quad \sup A = b,$$
$\max A$ does not exist.

The proof is very similar to the ones above, and is left to the reader for exercise. Let us now compute supremum and infimum of a set which is not an interval.

Proposition 4.12

Define the set
$$A := \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$
Then
$$\inf A = 0, \quad \sup A = \max A = 1.$$

> **Proof**
>
> *Part 1.* We have
> $$\frac{1}{n} \leq 1, \quad \forall n \in \mathbb{N}.$$
> Therefore 1 is an upper bound for A. Let us prove it is the least upper bound: let b be an upper bound for A. Since $1 \in A$ and b is an upper bound, we have $1 \leq b$. Hence 1 is the least upper bound, and
> $$\sup A = \max A = 1.$$
>
> *Part 2.* We have
> $$\frac{1}{n} > 0, \quad \forall n \in \mathbb{N},$$
> showing that 0 is a lower bound for A. Suppose by contradiction that 0 is not the infimum. Therefore 0 is not the largest lower bound. Then there exists $\varepsilon \in \mathbb{R}$ such that:
>
> - ε is a lower bound for A, that is,
> $$\varepsilon \leq \frac{1}{n}, \quad \forall n \in \mathbb{N}, \tag{4.13}$$
>
> - ε is larger than 0:
> $$0 < \varepsilon.$$
>
> As $\varepsilon > 0$, by the Archimedean Property there exists $n_0 \in \mathbb{N}$ such that
> $$0 < \frac{1}{n_0} < \varepsilon.$$
> This contradicts (4.13). Thus 0 is the largest lower bound of A, that is, $0 = \inf A$.
> *Part 3.* We have that $\min A$ does not exist. Indeed suppose by contradiction that $\min A$ exists. Then
> $$\min A = \inf A.$$
> As $\inf A = 0$ by Part 2, we conclude $\min A = 0$. As $\min A \in A$, we obtain $0 \in A$, which is a contradiction.

4.4 Density of $\mathbb{Q}$ in $\mathbb{R}$

A set S is dense in $\mathbb{R}$ if the elements of S are arbitrarily close to the elements of $\mathbb{R}$.

Definition 4.13: Dense set

Let $S \subseteq \mathbb{R}$. We say that S is dense in $\mathbb{R}$ if for every $x \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$, there exist $q \in \mathbb{Q}$ such that

$$|x - q| < \varepsilon.$$

In other words, the above definition is saying that S and $\mathbb{R}$ are tightly knitted together. An equivalent definition of dense set is given below.

Remark 4.14

Let $S \subseteq \mathbb{R}$. They are equivalent:

- S is dense in $\mathbb{R}$.
- For every pair of numbers $x, y \in \mathbb{R}$ with $x < y$, there exists $s \in S$ such that

$$x < s < y.$$

We now prove that $\mathbb{Q}$ is dense in $\mathbb{R}$.

Theorem 4.15: Density of $\mathbb{Q}$ in $\mathbb{R}$

Let $x, y \in \mathbb{R}$, with $x < y$. There exists $q \in \mathbb{Q}$ such that

$$x < q < y.$$
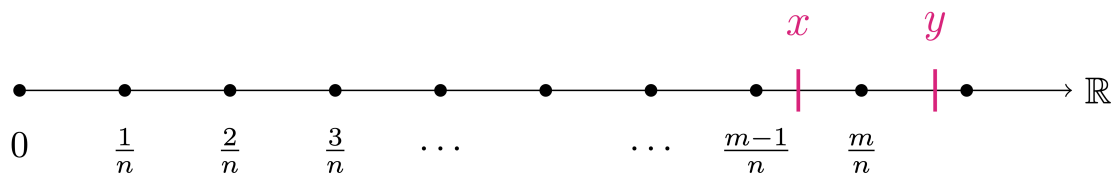
Figure 4.6: Let $n \in \mathbb{N}$ be such that $1/n < y - x$. Then take m so that $m/n \in (x, y)$.

Proof

We need to find $q \in \mathbb{Q}$ such that
$$x < q < y. \tag{4.14}$$
By definition of $\mathbb{Q}$, we have that q has to be $q = m/n$ for $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. Therefore (4.14) is equivalent to finding $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that
$$x < \frac{m}{n} < y. \tag{4.15}$$
The idea is to proceed as in Figure 4.6: We take n such that $1/n$ is small enough so that we can make m

jumps of size $1/n$ and end up between x and y.
To this end, let $n \in \mathbb{N}$ be such that

$$\frac{1}{n} < y - x. \tag{4.16}$$

Such n exists thanks to the Archimedean Property in Theorem 4.11 Point 2. Inequality (4.15) is equivalent to

$$nx < m < ny.$$

Take $m \in \mathbb{Z}$ such that

$$m - 1 \le nx < m. \tag{4.17}$$

> Why does such m exist? Because by Archimedean Property in Theorem 4.11 Point 1, there exists $m' \in \mathbb{Z}$ such that
>
> $$m' > nx$$
>
> We can then choose m to be the smallest element in $\mathbb{Z}$ such that $m > nx$. Such m satisfies (4.17).

The second inequality in (4.17) implies

$$x < \frac{m}{n},$$

which is the first inequality in (4.15). Now note that (4.16) is equivalent to $x < y - 1/n$. We can use the latter and the first inequality in (4.17) to estimate

$$m \le 1 + nx$$
$$< 1 + n\left(y - \frac{1}{n}\right)$$
$$= ny,$$

> which yields

$$\frac{m}{n} < y.$$

Therefore the second inequality in (4.15) is proven, concluding the proof.

We have constructed the real numbers $\mathbb{R}$ so that they would fill the **gaps** of $\mathbb{Q}$. Formally, these gaps are the numbers in $\mathbb{R} \setminus \mathbb{Q}$. Let us give a name to this set.

Definition 4.16: Irrational numbers

The set of irrational numbers in $\mathbb{R}$ is

$$\mathscr{I} := \mathbb{R} \setminus \mathbb{Q}.$$

> **Question 4.17**
>
> How many gaps does $\mathbb{Q}$ have? In other words, how many irrational numbers are out there?

The answer is quite surprising, and is a corollary of the density result of Theorem 4.15: The irrational numbers are dense in $\mathbb{R}$.

> **Corollary 4.18**
>
> Let $x, y \in \mathbb{R}$, with $x < y$. There exists $t \in \mathscr{I}$ such that
>
> $$x < t < y.$$

> **Proof**
>
> Consider
> $$\tilde{x} := x - \sqrt{2}, \quad \tilde{y} := y - \sqrt{2}.$$
> Since $x < y$, we have
> $$\tilde{x} < \tilde{y}.$$
> By Theorem 4.15 there exists $q \in \mathbb{Q}$ such that
> $$\tilde{x} < q < \tilde{y}.$$
> Adding $\sqrt{2}$ to the above inequalities we get
> $$x < t < y, \quad t := q + \sqrt{2}. \tag{4.18}$$
> We claim that $t \in \mathscr{I}$. Indeed, suppose by contradiction $t \in \mathbb{Q}$. Then
> $$\sqrt{2} = t - q \in \mathbb{Q},$$
> since $t, q \in \mathbb{Q}$, and $\mathbb{Q}$ is closed under summation. Since $\sqrt{2} \in \mathscr{I}$, we obtain a contradiction. Thus $t \in \mathscr{I}$ and (4.18) is our thesis.

4.5 Existence of k-th Roots

We have started our discussion by proving that

$$\sqrt{2} \notin \mathbb{Q}. \tag{4.19}$$

We have shown that (4.19) implies that the set

$$A := \{q \in \mathbb{Q} : q^2 < 2\}$$

does not have a supremum in $\mathbb{Q}$. We then introduced the Real Numbers $\mathbb{R}$ so that each non-empty and bounded above set would have a supremum. As the set A is non-empty and bounded above, there exists $\alpha \in \mathbb{R}$ such that

$$\alpha = \sup A.$$

We are going to prove that

$$\alpha^2 = 2,$$

which means that in $\mathbb{R}$ we can take the square root of 2. More in general, with the same fudamental idea, we can prove that for each $x \in \mathbb{R}$ with $x \geq 0$ and $k \in \mathbb{N}$, there exists $\alpha \in \mathbb{R}$ such that

$$\alpha^k = x.$$

Theorem 4.19: Existence of k-th roots

Let $x \in \mathbb{R}$ with $x \geq 0$ and $k \in \mathbb{N}$. There exists a unique $\alpha \in \mathbb{R}$ such that

$$\alpha^k = x.$$

The proof of Theorem 4.19 rests on similar ideas to the ones used to prove that $\mathbb{Q}$ does not have the cut property.

Proof: Proof of Theorem 4.19

Part 1: Uniqueness.
Suppose $\alpha_1, \alpha_2 \in \mathbb{R}$ are such that

$$\alpha_1^k = \alpha_2^k = x.$$

If $\alpha_1 \neq \alpha_2$, then

$$\alpha_1^k \neq \alpha_2^k,$$

obtaining a contradiction. Therefore $\alpha_1 = \alpha_2$.
Part 2: Existence.
Let $x \in \mathbb{R}$ with $x \geq 0$. If $x = 0$ there is nothing to prove, as

$$0^k = 0,$$

so that $\alpha = 0$. Therefore we can assume $x > 0$. Define the subset of $\mathbb{R}$

$$A := \{t \in \mathbb{R} : t^k < x\}.$$

Clearly A is non-empty and bounded above.

An upper bound is given, for example, by $b := x + 1$. Indeed, since we are assuming $x \geq 0$, then $x + 1 \geq 1$. In particular we have

$$(x + 1)^k \geq x + 1.$$

Let $t \in A$. Then

$$t^k < x < x + 1 < (x + 1)^k,$$

showing that $t < x + 1$.

By the Axiom of Completeness of $\mathbb{R}$, there exists $\alpha \in \mathbb{R}$ such that

$$\alpha = \sup A.$$

We claim that

$$\alpha^k = x. \tag{4.20}$$

Suppose by contradiction that (4.20) does not hold. We will need the formula: For all $a, b \in \mathbb{R}$ it holds

$$a^k - b^k = (a - b)(a^{k-1} + a^{k-2}b + a^{k-3}b^k + \ldots + ab^{k-2} + b^{k-1}). \tag{4.21}$$

Formula (4.21) can be easily proven by induction on k. Since we are assuming that (4.20) does not hold, we have two cases:

- $\alpha^k < x$: We know that α is the supremum of A. We would like to violate this, by finding a number L which is larger than α, but still belongs to A. This means L has to satisfy

$$\alpha < L, \quad L^k < x.$$

  We look for L of the form

$$L_n := \alpha + \frac{1}{n}$$

  for $n \in \mathbb{N}$ to be chosen later. Clearly

$$\alpha < L_n, \tag{4.22}$$

  for all $n \in \mathbb{N}$. We now search for $n_0 \in \mathbb{N}$ such that

$$L_{n_0}^k < x.$$

  Using formula (4.21) with $a = \alpha$ and $b = L_{n_0}$ we obtain

$$L_{n_0}^k - \alpha^k = \frac{1}{n_0}\left(L_{n_0}^{k-1} + L_{n_0}^{k-2}\alpha + \ldots + L_{n_0}\alpha^{k-2} + \alpha^{k-1}\right). \tag{4.23}$$

  Now notice that (4.22) implies

$$\alpha^j < L_{n_0}^j$$

for all $j \in \mathbb{N}$. Using this estimate on all the terms α^j appearing in the RHS of (4.23) we obtain

$$
\begin{aligned}
L_{n_0}^k - \alpha^k &= \frac{1}{n_0} \left(L_{n_0}^{k-1} + L_{n_0}^{k-2}\alpha + \ldots + L_{n_0}\alpha^{k-2} + \alpha^{k-1} \right) \\
&< \frac{1}{n_0} \left(L_{n_0}^{k-1} + L_{n_0}^{k-2}L_{n_0} + \ldots + L_{n_0}L_{n_0}^{k-2} + L_{n_0}^{k-1} \right) \\
&= \frac{k}{n_0} L_{n_0}^{k-1}
\end{aligned}
$$

Rearranging the above we get

$$
L_{n_0}^k < \frac{k}{n_0} L_{n_0}^{k-1} + \alpha^k \, . \tag{4.24}
$$

Now note that

$$
L_{n_0} = \alpha + \frac{1}{n_0} < \alpha + 1 \, .
$$

Therefore

$$
L_{n_0}^{k-1} < (\alpha + 1)^{k-1} \, ,
$$

and from (4.24) we obtain

$$
L_{n_0}^k < \frac{k}{n_0} (\alpha + 1)^{k-1} + \alpha^k \, .
$$

We wanted to find $n_0 \in \mathbb{N}$ so that $L_{n_0}^k < x$. Therefore we impose

$$
\frac{k}{n_0} (\alpha + 1)^{k-1} + \alpha^k < x \, ,
$$

and find that the above is satisfied for

$$
n_0 > \frac{k(\alpha + 1)^{k-1}}{x - \alpha^k} \, . \tag{4.25}
$$

Notice that the RHS in (4.25) is a positive real number, since $\alpha^k < x$ by assumption. Therefore, by the Archimedean Property of Theorem 4.11 Point 1, there exists $n_0 \in \mathbb{N}$ satisfying (4.25).
We have therefore shown the existence of $n_0 \in \mathbb{N}$ such that

$$
\alpha < L_{n_0} \, , \quad L_{n_0}^k < x \, .
$$

The above says that $L_{n_0} \in A$ and that

$$
L_{n_0} > \alpha = \sup A \, ,
$$

which is a contradiction, as $\sup A$ is an upper bound for A.

- $\alpha^k > x$: We know that α is the supremum of A. We would like to find a contradiction, by finding an upper bound L for A which is smaller than α. This means L has to satisfy

$$
L < \alpha \, , \quad L^k > x \, .
$$

Such L is an upper bound for A: If $t \in A$ then

$$t^k < x < L^k \quad \implies \quad t < L \,.$$

We therefore look for L of the form

$$L_n := \alpha - \frac{1}{n}$$

for $n \in \mathbb{N}$ to be chosen later. In this way

$$L_n < \alpha \,, \tag{4.26}$$

for all $n \in \mathbb{N}$. We now search for $n_0 \in \mathbb{N}$ such that

$$L_{n_0}^k > x \,.$$

Using formula (4.21) with $a = L_{n_0}$ and $b = \alpha$ we obtain

$$\alpha^k - L_{n_0}^k = \frac{1}{n_0} \left(\alpha^{k-1} + \alpha^{k-2} L_{n_0} + \ldots + \alpha L_{n_0}^{k-2} + L_{n_0}^{k-1} \right) \,. \tag{4.27}$$

Now notice that (4.26) implies

$$L_{n_0}^j < \alpha^j$$

for all $j \in \mathbb{N}$. Using this estimate on all the terms $L_{n_0}^j$ appearing in the RHS of (4.27) we obtain

$$\begin{aligned} \alpha^k - L_{n_0}^k &= \frac{1}{n_0} \left(\alpha^{k-1} + \alpha^{k-2} L_{n_0} + \ldots + \alpha L_{n_0}^{k-2} + L_{n_0}^{k-1} \right) \\ &< \frac{1}{n_0} \left(\alpha^{k-1} + \alpha^{k-2} \alpha + \ldots + \alpha \alpha^{k-2} + \alpha^{k-1} \right) \\ &= \frac{k}{n_0} \alpha^{k-1} \end{aligned}$$

Rearranging the above we get

$$L_{n_0}^k > \alpha^k - \frac{k}{n_0} \alpha^{k-1} \,.$$

We wanted to find $n_0 \in \mathbb{N}$ so that $L_{n_0}^k > x$. Therefore we impose

$$\alpha^k - \frac{k}{n_0} \alpha^{k-1} > x \,,$$

and find that the above is satisfied for

$$n_0 > \frac{k \alpha^{k-1}}{\alpha^k - x} \,. \tag{4.28}$$

Notice that the RHS in (4.28) is a positive real number, since $\alpha^k > x$ by assumption. Therefore, by the Archimedean Property of Theorem 4.11 Point 1, there exists $n_0 \in \mathbb{N}$ satisfying (4.28). We have therefore shown the existence of $n_0 \in \mathbb{N}$ such that

$$L_{n_0} < \alpha, \quad L_{n_0}^k > x.$$

Condition $L_{n_0}^k > x$ says that L_{n_0} is an upper bound for A. At the same time it holds

$$L_{n_0} < \alpha = \sup A,$$

which is a contradiction, as $\sup A$ is the smallest upper bound for A.

Therefore, both cases $\alpha^k > x$ and $\alpha^k < x$ lead to a contradiction. Hence $\alpha^k = x$, concluding.

Definition 4.20: k-th root of a number

Let $x \in \mathbb{R}$ with $x \geq 0$ and $k \in \mathbb{N}$. The real number α such that

$$\alpha^k = x$$

is called the k-**th root** of x, and is denoted by

$$\sqrt[k]{x} := \alpha.$$

4.6 Cardinality

We have proven that the sets or rational numbers $\mathbb{Q}$ and irrational numbers $\mathscr{I}$ are both dense in $\mathbb{R}$, with

$$\mathbb{R} = \mathbb{Q} \cup \mathscr{I}.$$

From this result we might think that $\mathbb{R}$ is obtained by mixing $\mathbb{Q}$ and $\mathscr{I}$ in equal proportions. This is however false. We will see that $\mathbb{R}$ has much more elements than $\mathbb{Q}$. Therefore also the set of irrational numbers $\mathscr{I}$ is much larger than $\mathbb{Q}$.

To make the above discussion precise, we need to define what we mean by size of a set. For this, we need the concept of bijective function.

Definition 4.21: Bijective function

Let X, Y be sets and $f : X \to Y$ be a function. We say that:

- f is **injective** if it holds:
$$f(x) = f(y) \quad \implies \quad x = y \,.$$

- f is **surjective** if it holds:
$$\forall y \in Y, \ \exists x \in X \ \text{s.t.} \ f(x) = y \,.$$

- f is **bijective** if it is both **injective** and **surjective**.

In other words: A function $f : X \to Y$ is

- injective if any two different elements in X are mapped into two different elements in Y.
- surjective if every element in Y has at least one element in X associated via f.
- bijective if to each element in X we associate one and only one element in Y via f.

Example 4.22: Injectivity

Consider the sets
$$X = \{1, 2, 3\}, \quad Y = \{a, b, c, d, e\} \,.$$

- The function $f : X \to Y$ defined by
$$f(1) = c, \quad f(2) = a, \quad f(3) = e \,,$$
is injective.

- The function $g : X \to Y$ defined by
$$g(1) = c, \quad g(2) = a, \quad g(3) = c \,,$$
is not injective, since
$$g(1) = g(3) = c, \quad 1 \neq 3 \,.$$

- The function $h : \mathbb{R} \to \mathbb{R}$ defined by
$$h(x) = x^2$$
is not injective, since
$$h(1) = h(-1) = 1, \quad 1 \neq -1 \,.$$

- The function $l : \mathbb{R} \to \mathbb{R}$ defined by
$$l(x) = 2x$$
is injective, since
$$l(x) = l(y) \quad \implies \quad 2x = 2y \quad \implies \quad x = y \,.$$

Example 4.23: Surjectivity

Consider the sets
$$X = \{1, 2, 3, 4\}, \quad Y = \{a, b, c\}.$$

- The function $f: X \to Y$ defined by
$$f(1) = c, \quad f(2) = a, \quad f(3) = a, \quad f(4) = b,$$
is surjective.

- The function $g: X \to Y$ defined by
$$g(1) = a, \quad g(2) = a, \quad g(3) = c, \quad g(4) = a,$$
is not surjective, since there is no element $x \in X$ such that
$$g(x) = b.$$

- The function $h: \mathbb{R} \to \mathbb{R}$ defined by
$$h(x) = x^2$$
is not surjective, since there is no $x \in \mathbb{R}$ such that
$$h(x) = x^2 = -1.$$

- The function $l: \mathbb{R} \to [0, \infty)$ defined by
$$l(x) = x^2$$
is surjective, since for every $y \geq 0$ there exists $x \in \mathbb{R}$ such that
$$l(x) = x^2 = y.$$

This is true by Theorem 4.19.

Example 4.24: Bijectivity

- Let $X = \{1, 2, 3\}, Y = \{a, b, c\}$. The function $f: X \to Y$ defined by
$$f(1) = c, \quad f(2) = a, \quad f(3) = b,$$
is bijective, since it is both injective and surjective.

- Let $X = \{1, 2, 3\}, Y = \{a, b\}$. The function $g: X \to Y$ defined by
$$g(1) = a, \quad g(2) = b, \quad g(3) = b,$$
is not bijective, since it is not injective: we have
$$g(2) = g(3) = b \quad 2 \neq 3.$$

- Let $X = \{1, 2\}, Y = \{a, b, c\}$. The function $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(1) = a, \quad h(2) = c,$$

  is not bijective, since it is not surjective: there is no $x \in X$ such that

$$h(x) = b.$$

- Let $X = \{1, 2, 3\}, Y = \{a, b, c\}$. The function $l : \mathbb{R} \to [0, \infty)$ defined by

$$l(1) = a, \quad l(2) = a, \quad l(3) = b,$$

  is not bijective, as it is neither injective nor surjective.

We are ready to define the size of a set.

Definition 4.25:   Cardinality, finite, countable, uncountable

Let X be a set. The **cardinality** of X is the number of elements in X. We denote the cardinality of X by

$$|X| := \# \text{ of elements in } X.$$

Further, we say that:

- X is **finite** if there exists a natural number $n \in \mathbb{N}$ and a bijection

$$f : X \to \{1, 2, \dots, n\}.$$

  In particular
$$|X| = n.$$

- X is **countable** if there exists a bijection

$$f : X \to \mathbb{N}.$$

  In this case we denote the cardinality of X by

$$|X| = |\mathbb{N}|.$$

- X is **uncountable** if X is neither finite, nor countable.

In other words: A set X is

- finite, if X can be listed as

$$X = \{x_1, \dots, x_n\}$$

for some $n \in \mathbb{N}$.

- countable, if X can be listed as

$$X = \{x_n \; : \; n \in \mathbb{N}\}.$$

- uncountable, if X cannot be listed.

Question 4.26

Is there an intermediate cardinality between finite and countable?

The answer is no, as shown in the next proposition.

Proposition 4.27

Let X be a countable set and $A \subseteq X$. Then either A is finite or countable.

Proof

If A is finite we are done. Therefore suppose A is not finite. Since X is countable there exists a bijection $f : \mathbb{N} \to X$. Let $n_1 \in \mathbb{N}$ be such that

$$n_1 = \min\{n \in \mathbb{N} \;\; \text{s.t.} \;\; f(n) \in A\}.$$

Note that n_1 exists since f is surjective. Define

$$g(1) := f(n_1).$$

Now let
$$n_2 = \min\{n \in \mathbb{N} \;\; \text{s.t.} \;\; n > n_1, \;\; f(n) \in A\}.$$

Notice that n_2 exists, since f is surjective and A is not finite. Set

$$g(2) := f(n_2).$$

Iterating, we define
$$n_k = \min\{n \in \mathbb{N} \;\; \text{s.t.} \;\; n > n_{k-1}, \;\; f(n) \in A\}$$

and
$$g(k) := f(n_k).$$

In this way we have defined a function $g : \mathbb{N} \to A$. We have:

- g is injective: This is because g was defined thourgh f, and f is injective.

- g is surjective: If $x \in A$, by surjectivity of f there exists $\tilde{n} \in \mathbb{N}$ such that $f(\tilde{n}) = x$. Therefore

$$\tilde{n} \in \{n \in \mathbb{N} \ \text{s.t.} \ f(n) \in A\},$$

so that $n_k = \tilde{n}$, for some $k \geq \tilde{n} - 1$.

Hence g is bijective, showing that A is countable.

Example 4.28

1. Let $X = \{a, b, c\}$ and $Y = \{1, 2, 3\}$. The function $f : X \to Y$ defined by

$$f(1) = a, \quad f(2) = b, \quad f(3) = c,$$

is bijective. Therefore X is finite, with $|X| = 3$.

2. Let $X = \mathbb{N}$. The function $f : X \to \mathbb{N}$ defined by

$$f(n) := n,$$

is bijective. Therefore $X = \mathbb{N}$ is countable.

3. Let X be the set of even numbers
$$X = \{2n \ : \ n \in \mathbb{N}\}.$$

Define the map $f : X \to \mathbb{N}$ by
$$f(m) := \frac{m}{2}.$$

We have that:

- f is injective: $f(m) = f(k)$ implies that $m/2 = k/2$ which implies $m = k$.
- f is surjective: If $n \in \mathbb{N}$, then $f(2n) = n$.

Therefore f is bijective, showing that X is countable and $|X| = |\mathbb{N}|$.

4. Let $X = \mathbb{Z}$ the set of integers. Define $f : \mathbb{N} \to \mathbb{Z}$ by

$$f(n) := \begin{cases} \dfrac{n}{2} & \text{if } n \text{ even} \\ -\dfrac{n+1}{2} & \text{if } n \text{ odd} \end{cases}$$

For example

$$f(0) = 0, \quad f(1) = -1, \quad f(2) = 1, \quad f(3) = -2,$$
$$f(4) = 2, \quad f(5) = -3, \quad f(6) = 3, \quad f(7) = -4.$$

We have:

- f is injective: Indeed, suppose that $m \neq n$. If n and m are both even or both odd we have, respectively

$$f(m) = \frac{m}{2} \neq \frac{n}{2} = f(n)$$

$$f(m) = -\frac{m+1}{2} \neq -\frac{n+1}{2} = f(n).$$

If instead m is even and n is odd, we get

$$f(m) = \frac{m}{2} \neq -\frac{n+1}{2} = f(n),$$

since the LHS is positive and the RHS is negative. The case when m is odd and n even is similar.

- f is surjective: Let $z \in \mathbb{Z}$. If $z \geq 0$, then $m := 2z$ belongs to $\mathbb{N}$, is even, and

$$f(m) = f(2z) = z.$$

If instead $z < 0$, then $m := -2z - 1$ belongs to $\mathbb{N}$, is odd, and

$$f(m) = f(-2z - 1) = z.$$

Therefore f is bijective, showing that $\mathbb{Z}$ is countable and

$$|\mathbb{Z}| = |\mathbb{N}|.$$

We have seen that the sets $\mathbb{N}$ and $\mathbb{Z}$ are countable. What about $\mathbb{Q}$? To study this case, we need the following result.

Proposition 4.29

Let the set A_n be countable for all $n \in \mathbb{N}$. Define

$$A = \bigcup_{n \in \mathbb{N}} A_n.$$

Then A is countable.

Proof

Since each A_i is countable, we can list their elements as

$$A_i = \{a_k^i \ : \ k \in \mathbb{N}\} = \{a_1^i, \, a_2^i, \, a_3^i, \, a_4^i, \, ...\}.$$

The proof that A is countable is based on a diagonal argument by Georg Cantor, see Wikipedia page. The idea of th is that we can list the elements of the sets A_i in an infinite square: In the first row we put the elements of A_1, in the second row the elements of A_2, and so on. Therefore the i-th row contains the elements of A_i. This procedure is illustrated in Figure 4.7. Therefore this infinite square contains all the elements of A. We then list all the elements of the square by looking at the diagonals, as shown in Figure 4.7. This procedure defines a function $f : \mathbb{N} \to A$. For example the first few terms of f are

$$f(1) = a_1^1, \quad f(2) = a_2^1, \quad f(3) = a_1^2, \quad f(4) = a_1^3.$$

Since f is injective and surjective, we have that A is countable.
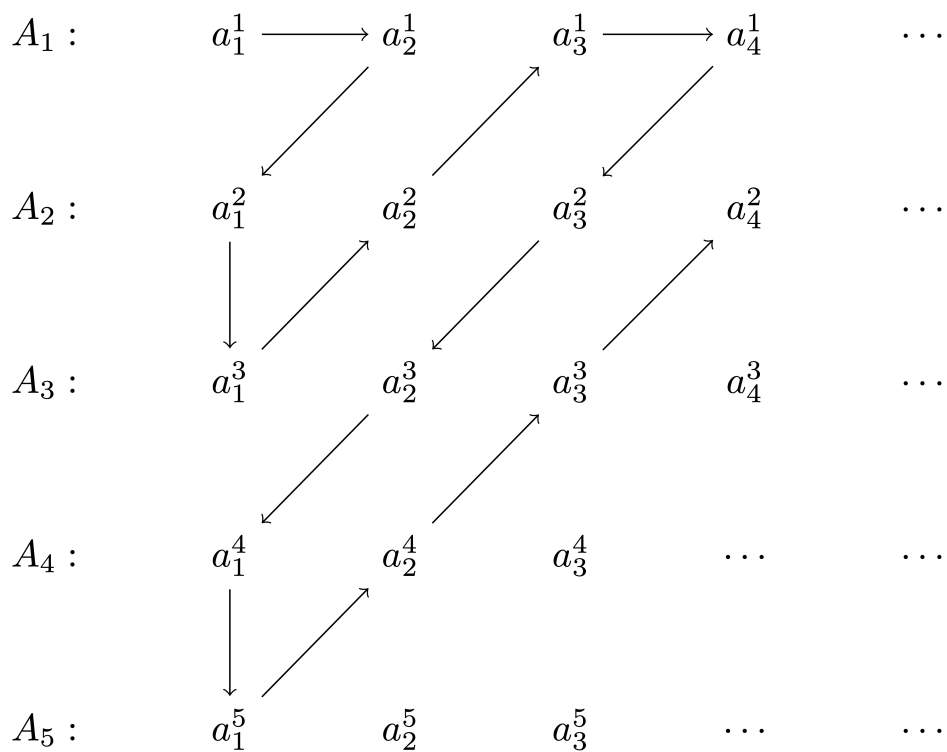
Figure 4.7: The i-th row contains all the elements $a_1^i, a_2^i, a_3^i, ...$ of the countable set A_i. We define the function $f : \mathbb{N} \to A$ by going throgh the square diagonally.

Theorem 4.30: $\mathbb{Q}$ is countable

The set of rational numbers $\mathbb{Q}$ is countable.

Proof

For $i \in \mathbb{N}$ define the sets

$$L_i := \left\{ \frac{m}{i} \ : \ m \in \mathbb{Z} \right\}.$$

We have that $f : L_i \to \mathbb{Z}$ defined by

$$f\left(\frac{m}{i}\right) := m$$

is a bijection. As $\mathbb{Z}$ is countable, we deduce that L_i is countable. Therefore the set L defined by

$$L := \bigcup_{i \in \mathbb{N}} L_i$$

is countable by Proposition 4.29. Clearly we have

$$\mathbb{Q} \subseteq L.$$

Since $\mathbb{Q}$ is not finite, by Proposition 4.27 we conclude that $\mathbb{Q}$ is countable.

We have proven that the sets

$$\mathbb{N}, \quad \mathbb{Z}, \quad \mathbb{Q},$$

are all countable. What about $\mathbb{R}$?

Theorem 4.31: $\mathbb{R}$ is uncountable

The set of Real Numbers $\mathbb{R}$ is **uncountable**.

Proof

Suppose by contradiction $\mathbb{R}$ is countable. Then there exists a bijection $f : \mathbb{N} \to \mathbb{R}$, meaning that we can list the elements of $\mathbb{R}$ as

$$\mathbb{R} = \{x_1, x_2, x_3, x_4, x_5, \ldots\}.$$

Let I_1 be a closed interval such that

$$x_1 \notin I_1.$$

Let I_2 be another closed interval, contained in I_1, and such that $x_2 \notin I_2$. Such interval exists, because I_1 contains two disjoint closed intervals: hence x_2 can be at most in one of these two intervals. To

summarize, we have

$$x_1 \notin I_1 , \quad x_2 \notin I_2 , \quad I_2 \subseteq I_1 .$$

We can iterate this procedure, and construct a sequence of nested intervals I_n such that

$$I_{n+1} \subseteq I_n , \quad x_n \notin I_n ,$$

for all $n \in \mathbb{N}$, see Figure 4.8. Since $x_k \notin I_k$, we conclude that

$$x_k \notin \bigcap_{n=1}^{\infty} I_n , \quad \forall k \in \mathbb{N} .$$

As the points x_k are all the elements of $\mathbb{R}$, we conclude that

$$\bigcap_{n=1}^{\infty} I_n = \emptyset .$$

However the Nested Interval Property of Theorem 4.14 implies that

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset ,$$

yielding a contradiction. Therefore $\mathbb{R}$ is uncountable.
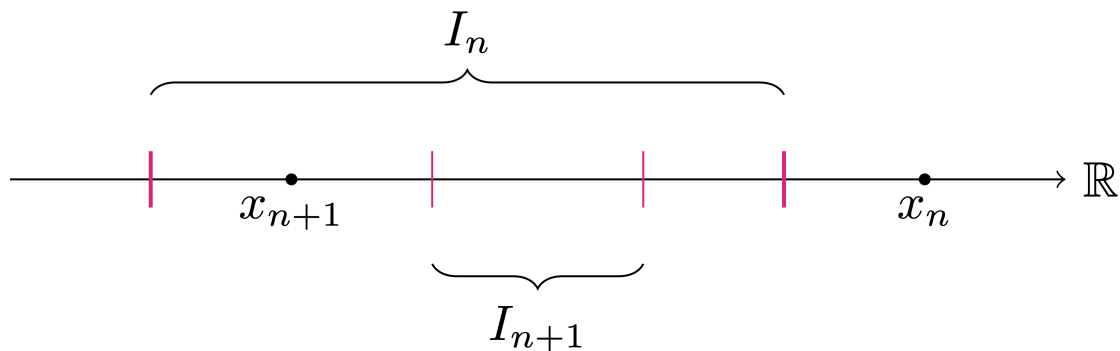
Figure 4.8: The intervals I_n are nested, and can be chosen so that $x_n \notin I_n$.

As a corollary we obtain that also the irrational numbers are uncountable.

Theorem 4.32

The set of irrational numbers
$$\mathscr{I} := \mathbb{R} \setminus \mathbb{Q}$$
is uncountable.

Proof

In Theorems 4.30, 4.31 we have shown that $\mathbb{R}$ in uncountable and $\mathbb{Q}$ is countable. Suppose by contradiction that $\mathscr{I}$ is countable. Then
$$\mathbb{Q} \cup \mathscr{I}$$
is countable by Proposition 4.29, being union of countable sets. Since by definition
$$\mathbb{R} = \mathbb{Q} \cup \mathscr{I} \,,$$
we conclude that $\mathbb{R}$ is countable. Contradiction.

5 Complex Numbers

We have seen that $\sqrt{x}$ exists in $\mathbb{R}$ for all for $x \geq 0$. We defined

$$\sqrt{x} := \alpha, \quad \alpha := \sup\{t \in \mathbb{R} : t^2 < x\},$$

and proved that

$$\alpha^2 = x.$$

This procedure was possible for $x \geq 0$.

Question 5.1

Is there a number $\alpha \in \mathbb{R}$ such that

$$\alpha^2 = -1 \ ? \tag{5.1}$$

The answer to the above question is no. This is because $\mathbb{R}$ is an ordered field, and from axiom (MO) it follows that:

$$x^2 \geq 0, \quad \forall\, x \in \mathbb{R}.$$

However we would still like to solve equation (5.1) somehow. To do this, we introduce the **imaginary numbers** or **complex numbers**. We define i to be that *number* such that

$$i^2 = -1.$$

Formally, we can also think of $i = \sqrt{-1}$. We can use this speacial number to define the square root of a negative number $x < 0$:

$$\sqrt{x} := i\sqrt{-x}.$$

Note that $\sqrt{-x}$ is properly defined in $\mathbb{R}$, because $-x > 0$ if $x < 0$.

5.1 The field $\mathbb{C}$

We would like to be able to do calculations with the newly introduced complex numbers, and investigate their properties. We can introduce them rigorously as a field, as we did for $\mathbb{R}$.

Definition 5.2: Complex Numbers

The set of complex numbers $\mathbb{C}$ is defined as

$$\mathbb{C} := \mathbb{R} \oplus i\mathbb{R} := \{x \oplus iy : \ x, y \in \mathbb{R}\}.$$

In the above the symbol $\oplus$ is used to denote the pair

$$x \oplus iy = (x, y)$$

with $x, y \in \mathbb{R}$. This means that x and y play different *roles*.

Definition 5.3

For a complex number

$$z = x \oplus iy \in \mathbb{C}$$

we say that

- x is the **real part** of z, and denote it by

$$x = \text{Re}(z)$$

- y is the **imaginary part** of z, and denote it by

$$y = \text{Im}(z)$$

We say that

- If $\text{Re} \, z = 0$ then z is a **purely imaginary** number.
- If $\text{Im} \, z = 0$ then z is a **real** number.

In order to make the set $\mathbb{C}$ into a field, we first have to define the two binary operations of addition $+$ and multiplication $\cdot$,

$$+, \cdot \ : \ \mathbb{C} \times \mathbb{C} \to \mathbb{C}.$$

Then we need to prove that these operations satisfy all the field axioms.

Definition 5.4: Addition in $\mathbb{C}$

Let $z_1, z_2 \in \mathbb{C}$, so that

$$z_1 = x_1 \oplus iy_1 \,, \quad z_2 = x_2 \oplus iy_2 \,,$$

for some $x_1, x_2, y_1, y_2 \in \mathbb{R}$. We define the sum of z_1 and z_2 as

$$z_1 + z_2 = (x_1 \oplus iy_1) + (x_2 \oplus iy_2) := (x_1 + x_2) \oplus i(y_1 + y_2)$$

where the $+$ symbol on the right hand side is the addition operator in $\mathbb{R}$.

Clearly, $z_1 + z_2$ as defined above is an element of $\mathbb{C}$. Therefore $+$ defines a binary operation over $\mathbb{C}$.

Notation 5.5

From the above definition, we have that, for all $x, y \in \mathbb{R}$,

$$(x \oplus i0) + (0 \oplus iy) = x \oplus iy.$$

To simplify notation, we will write

$$x \oplus i0 = x, \qquad 0 \oplus iy = iy$$

and

$$x \oplus iy = x + iy.$$

We will also often swap i and y, writing equivalently

$$x + iy = x + yi.$$

We now want to define multiplication between complex numbers.

Remark 5.6: Formal calculation for multiplication in $\mathbb{C}$

How to define multiplication in $\mathbb{C}$? Whatever the definition may be, at least it has to give that that

$$i^2 = i \cdot i = -1.$$

Keeping the above in mind, let us do some formal calculations: For $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$ we have

$$
\begin{aligned}
z_1 \cdot z_2 &= (x_1 + iy_1) \cdot (x_2 + iy_2) \\
&= x_1 \cdot x_2 + x_1 \cdot iy_2 + x_2 \cdot iy_1 + y_1 \cdot i^2 y_2 \\
&= (x_1 \cdot x_2 - y_1 \cdot y_2) + i(x_1 \cdot y_2 + x_2 \cdot y_1)
\end{aligned}
$$

Remark 5.6 motivates the following definition of multiplication.

Definition 5.7: Multiplication in $\mathbb{C}$

Let $z_1, z_2 \in \mathbb{C}$, so that

$$z_1 = x_1 \oplus iy_1, \qquad z_2 = x_2 \oplus iy_2,$$

for some $x_1, x_2, y_1, y_2 \in \mathbb{R}$. We define the multiplication of z_1 and z_2 as

$$z_1 \cdot z_2 = (x_1 + iy_1) \cdot (x_2 + iy_2) := (x_1 \cdot x_2 - y_1 \cdot y_2) + i(x_1 \cdot y_2 + x_2 \cdot y_1),$$

---

where the operations $+$ and $\cdot$ on the right hand side are the operations in $\mathbb{R}$.

Clearly, $z_1 \cdot z_2$ as defined above is an element of $\mathbb{C}$. Therefore $\cdot$ defines a binary operation over $\mathbb{C}$.

Remark 5.8

To check that we have given a good definition of product, we should have that

$$i^2 = -1 \,,$$

as expected. Indeed:

$$i^2 = (0 + 1i) \cdot (0 + 1i)$$
$$= (0 \cdot 0 - 1 \cdot 1) + (0 \cdot 1 + 0 \cdot 1)i = -1 \,.$$

Important

In view of Remark 5.8, we see that he formal calculations in Remark 5.6 are compatible with the definition of multiplication of complex numbers. Therefore, it is not necessary to memorize the multiplication formula, but it suffices to carry out calculations as usual, and replace i^2 by -1.

Example 5.9

Suppose we want to multiply the complex numbers

$$z = -2 + 3i \,, \quad w = 1 - i \,.$$

Using the definition we compute

$$z \cdot w = (-2 + 3i) \cdot (1 - i)$$
$$= (-2 - (-3)) + (2 + 3)i$$
$$= 1 + 5i \,.$$

Alternatively, we can proceed formally as in Remark 5.6. We just need to recall that i^2 has to be replaced

---

with -1:

$$z \cdot w = (-2 + 3i) \cdot (1 - i)$$
$$= -2 + 2i + 3i - 3i^2$$
$$= (-2 + 3) + (2 + 3)i$$
$$= 1 + 5i.$$

We now want to check that

$$(\mathbb{C}, +, \cdot)$$

is a field. All the field axioms are trivial to check, except for the existence of additive and multiplicative inverses.

Proposition 5.10: Additive inverse in $\mathbb{C}$

The neutral element of addition in $\mathbb{C}$ is the number

$$0 := 0 + 0i.$$

For any $z = x + iy \in \mathbb{C}$, the unique additive inverse is given by

$$-z := -x - iy.$$

The proof is immediate and is left as an exercise. The multiplication requires more care.

Remark 5.11: Formal calculation for multiplicative inverse

Let us first carry our some formal calculations. Let

$$z = x + iy \in \mathbb{C}, \quad z \neq 0.$$

First, note that

$$z \cdot 1 = (x + iy) \cdot (1 + 0i) = x + iy = z,$$

and therefore 1 is the neutral element of multiplication. Thus, the inverse of z should be a complex number $z^{-1} \in \mathbb{C}$ such that

$$z \cdot z^{-1} = 1.$$

We would like to define

$$z^{-1} = \frac{1}{x + iy}.$$

Such number does not belong to $\mathbb{C}$, as it is not of the form $a + ib$ for some $a, b \in \mathbb{R}$. However it is what the inverse should look like. Proceeding formally:

$$
\begin{aligned}
\frac{1}{x + iy} &= \frac{1}{x + iy} \cdot 1 \\
&= \frac{1}{x + iy} \cdot \frac{x - iy}{x - iy} \\
&= \frac{x - iy}{x^2 - (iy)^2} \\
&= \frac{x - iy}{x^2 + y^2} \\
&= \frac{x}{x^2 + y^2} + i\,\frac{-y}{x^2 + y^2}\,.
\end{aligned}
$$

The right hand side is an element of $\mathbb{C}$, and looks like a good candidate for z^{-1}.

Motivated by the above remark, we define inverses in $\mathbb{C}$ in the following way.

Proposition 5.12:   Multiplicative inverse in $\mathbb{C}$

The neutral element of multiplication in $\mathbb{C}$ is the number

$$
1 := 1 + 0i\,.
$$

For any $z = x + iy \in \mathbb{C}$, the unique multiplicative inverse is given by

$$
z^{-1} := \frac{x}{x^2 + y^2} + i\,\frac{-y}{x^2 + y^2}\,.
$$

Proof

It is immediate to check that 1 is the neutral element of multiplication in $\mathbb{C}$. For the remaining part of the statement, set

$$
w := \frac{x}{x^2 + y^2} + i\,\frac{-y}{x^2 + y^2}\,.
$$

We need to check that $z \cdot w = 1$

$$z \cdot w = (x + iy) \cdot \left(\frac{x}{x^2 + y^2} + i \frac{-y}{x^2 + y^2} \right)$$

$$= \left(\frac{x^2}{x^2 + y^2} - \frac{y \cdot (-y)}{x^2 + y^2} \right) + i \left(\frac{x \cdot (-y)}{x^2 + y^2} + \frac{xy}{x^2 + y^2} \right)$$

$$= 1 \,,$$

so indeed $z^{-1} = w$.

Important

It is not necessary to memorize the formula for z^{-1}. Indeed one can just remember the trick of multiplying by

$$1 = \frac{x - iy}{x - iy} \,,$$

and proceed formally, as done in Remark 5.11.

Example 5.13

Let $z = 3 + 2i$. We want to compute z^{-1}. By the formula in Propostion 5.12 we immediately get

$$z^{-1} = \frac{3}{3^2 + 2^2} + \frac{-2}{3^2 + 2^2} i = \frac{3}{13} - \frac{2}{13} i \,.$$

Alternatively, we can proceed formally as in Remark 5.11

$$(3 + 2i)^{-1} = \frac{1}{3 + 2i}$$

$$= \frac{1}{3 + 2i} \frac{3 - 2i}{3 - 2i}$$

$$= \frac{3 - 2i}{3^2 + 2^2}$$

$$= \frac{3}{13} - \frac{2}{13} i \,,$$

and obtain the same result.

We can now prove that $\mathbb{C}$ is a field.

Theorem 5.14

$(\mathbb{C}, +, \cdot)$ is a field.

Proof

We need to check that all field axioms hold. For the addition we have

- (A1) To show that $+$ is commutative, note that

$$
\begin{aligned}
(x + iy) + (a + ib) &= (x + a) + i(y + b) \\
&= (a + x) + i(b + y) \\
&= (a + ib) + (x + iy),
\end{aligned}
$$

  where we used Definition 5.4 in the first and last equality, and the commutative property of the real numbers (which holds since by definition $\mathbb{R}$ is a field) in the second equality. Associativity can be checked in the same way.

- (A2) The neutral element of addition is 0, as stated in Proposition 5.10.

- (A3) Existence of additive inverses is given by Proposition 5.10.

For multiplication we have:

- (M1) Commutativity and associativity of product in $\mathbb{C}$ can be checked using Definition 5.7 and commutativity and associativity of sum and multiplication in $\mathbb{R}$.

- (M2) The neutral element of multiplication is 1, as stated in Propostion 5.12.

- (M3) Existence of multiplicative inverses is guaranteed by Proposition 5.12.

Finally one should check the associative property (AM). This is left as an exercise.

5.1.1 Division in $\mathbb{C}$

Suppose we want to divide two complex numbers $w, z \in \mathbb{C}, z \neq 0$, with

$$
z = x + iy, \quad w = a + ib.
$$

We have two options:

1. Use the formula for the inverse from Proposition 5.12 and compute

$$
z^{-1} := \frac{x}{x^2 + y^2} + i\,\frac{-y}{x^2 + y^2}\,.
$$

Then we use the multiplication formula of Definition 5.7 to compute

$$\frac{w}{z} = w \cdot z^{-1}$$

$$= (a + ib) \cdot \left(\frac{x}{x^2 + y^2} + i \frac{-y}{x^2 + y^2} \right)$$

$$= \frac{(ax + by) + i(bx - ay)}{x^2 + y^2}$$

2. Proceed formally as in Remark 5.11, using the multiplication by 1 trick. We would have

$$\frac{w}{z} = \frac{a + ib}{x + iy}$$

$$= \frac{a + ib}{x + iy} \frac{x - iy}{x - iy}$$

$$= \frac{(ax + by) + i(bx - ay)}{x^2 + y^2}$$

Example 5.15

Let $w = 1 + i$ and $z = 3 - i$. We compute $\frac{w}{z}$ using the two options we have:

1. Using the formula for the inverse from Proposition 5.12 we compute

$$z^{-1} = \frac{x}{x^2 + y^2} + i \frac{-y}{x^2 + y^2}$$

$$= \frac{3}{3^2 + 1^2} - i \frac{-1}{3^2 + 1^2}$$

$$= \frac{3}{10} + \frac{1}{10} i$$

and therefore

$$\frac{w}{z} = w \cdot z^{-1}$$

$$= (1 + i) \left(\frac{3}{10} + \frac{1}{10} i \right)$$

$$= \left(\frac{3}{10} - \frac{1}{10} \right) + \left(\frac{1}{10} + \frac{3}{10} \right) i$$

$$= \frac{2}{10} + \frac{4}{10} i$$

$$= \frac{1}{5} + \frac{2}{5} i$$

2. We proceed formally, using the multiplication by 1 trick. We have

$$
\begin{aligned}
\frac{w}{z} &= \frac{1+i}{3-i} \\
&= \frac{1+i}{3-i}\frac{3+i}{3+i} \\
&= \frac{3-1+(3+1)i}{3^2+1^2} \\
&= \frac{2}{10}+\frac{4}{10}i \\
&= \frac{1}{5}+\frac{2}{5}i
\end{aligned}
$$

5.1.2 $\mathbb{C}$ is not ordered

We have seen that $(\mathbb{C},+,\cdot)$ is a field. One might wonder whether $\mathbb{C}$ is also an ordered field. It turns out that this is not the case.

Theorem 5.16

The field $(\mathbb{C},+,\cdot)$ is not ordered.

Proof

Suppose that $\mathbb{C}$ is an ordered field, that is, there exists an order relation $\leq$ on $\mathbb{C}$ compatible with the operations $+$ and $\cdot$. By axiom (MO) it follows that for all elements $z \in \mathbb{C}, z \neq 0$, we have that $z^2 > 0$. But since $i^2 = -1 < 0$, we get a contradiction.

Hence, it is not possible to compare two complex numbers.

5.1.3 Completeness of $\mathbb{C}$

One might also wonder whether $\mathbb{C}$ is complete. Our definition of completeness uses the notion of supremum, which only makes sense if the field is ordered. This is not the case for $\mathbb{C}$ as we have seen in Theorem 5.16.

Still, it is possible to give a different definition of completeness using the notion of Cauchy sequence. In ordered fields, this new definition of completeness is equivalent to the definition which uses the supremum.

The new definition of completeness with Cauchy sequences also makes sense in non-ordered fields. We will see that $\mathbb{C}$ is a complete field, according to this new definition.

5.2 Complex conjugates

When computing inverses, we used the trick to multiply by 1:

$$z^{-1} = \frac{1}{z} \cdot 1 = \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy}.$$

The complex number $x - iy$ is obtained by changing the sign to the imaginary part of $z = x + iy$. We give a name to this operation.

Definition 5.17: Complex conjugate

Let $z = x + iy$. We call the **complex conjugate** of z, denoted by $\bar{z}$, the complex number

$$\bar{z} = x - iy.$$

Example 5.18

We have the following conjugates:

$$\overline{3+4i} = 3 - 4i, \qquad\qquad \overline{3-4i} = 3 + 4i,$$
$$\overline{-3+4i} = -3 - 4i, \qquad\qquad \overline{-3-4i} = -3 + 4i,$$
$$\overline{3} = 3, \qquad\qquad \overline{4i} = -4i.$$

Complex conjugates have the following properties:

Theorem 5.19

For all $z_1, z_2 \in \mathbb{C}$ it holds:

- $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

- $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$

Proof

Let $z_1, z_2 \in \mathbb{C}$. Then

$$z_1 = x_1 + iy_1, \qquad z_2 = x_2 + iy_2,$$

for some $x_1, y_1, x_2, y_2 \in \mathbb{R}$.

- Using the definition of addition in $\mathbb{C}$ and of conjugate,

$$
\begin{aligned}
\overline{z_1 + z_2} &= \overline{(x_1 + iy_1) + (x_2 + iy_2)} \\
&= \overline{(x_1 + x_2) + i(y_1 + y_2)} \\
&= (x_1 + x_2) - i(y_1 + y_2) \\
&= (x_1 - iy_1) + (x_2 - iy_2) \\
&= \overline{x_1 + iy_1} + \overline{x_2 + iy_2} \\
&= \overline{z_1} + \overline{z_2} \, .
\end{aligned}
$$

- Using the definition of multiplication in $\mathbb{C}$ and of conjugate,

$$
\begin{aligned}
\overline{z_1 \cdot z_2} &= \overline{(x_1 + iy_1) \cdot (x_2 + iy_2)} \\
&= \overline{(x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)} \\
&= (x_1 x_2 - y_1 y_2) - i(x_1 y_2 + x_2 y_1) \\
&= (x_1 - iy_1) \cdot (x_2 - iy_2) \\
&= \overline{z_1} \cdot \overline{z_2}
\end{aligned}
$$

Example 5.20

Let $z_1 = 3 - 4i$ and $z_2 = -2 + 5i$. Then

- Let us check that
$$
\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}
$$
Indeed, we have
$$
z_1 + z_2 = 1 + i \quad \Longrightarrow \quad \overline{z_1 + z_2} = 1 - i \, .
$$
On the other hand
$$
\overline{z_1} = 3 + 4i \, , \quad \overline{z_2} = -2 - 5i \quad \Longrightarrow \quad \overline{z_1} + \overline{z_2} = 1 - i \, .
$$

- Let us check that
$$
\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}
$$
Indeed,
$$
\begin{aligned}
\overline{z_1} \cdot \overline{z_2} &= (3 + 4i) \cdot (-2 + 5i) \\
&= (-6 + 20) + (8 + 15)i \\
&= 14 + 23i
\end{aligned}
$$

so that

$$\overline{z_1 \cdot z_2} = 14 - 23i$$

On the other hand:

$$\begin{aligned}
\overline{z_1} \cdot \overline{z_2} &= (3 + 4i) \cdot (-2 - 5i) \\
&= (-6 + 20) + (-15 - 8)i \\
&= 14 - 23i
\end{aligned}$$

5.3 The complex plane

We can depict a real number x as a point on the one-dimensional real line $\mathbb{R}$. The distance between two real numbers $x, y \in \mathbb{R}$ on the real line is given by $|x - y|$, see Figure 5.1.
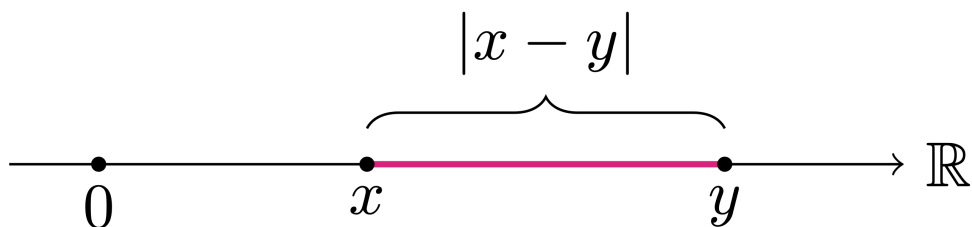
Figure 5.1: Two points x and y on the real line $\mathbb{R}$. Their distance is $|x - y|$.

We would like to do something similar for the complex numbers, but the point

$$z = x + iy, \quad x, y \in \mathbb{R}.$$

We therefore depict $z = z + iy$ in the two-dimensional plane at the point with (Cartesian) coordinates (x, y). This two-dimensional plane in which we can depict all complex numbers is called the **complex plane**. The origin of such plane, with coordinates $(0, 0)$, corresponds to the complex number

$$0 + 0i = 0,$$

see Figure 5.2.

5.3.1 Distance on $\mathbb{C}$

The Cartesian representation allows us to introduce a distance between two complex numbers. Let us start with the distance between a complex number $z = x + iy$ and 0. By Pythagoras Theorem this distance is given by

$$\sqrt{x^2 + y^2},$$
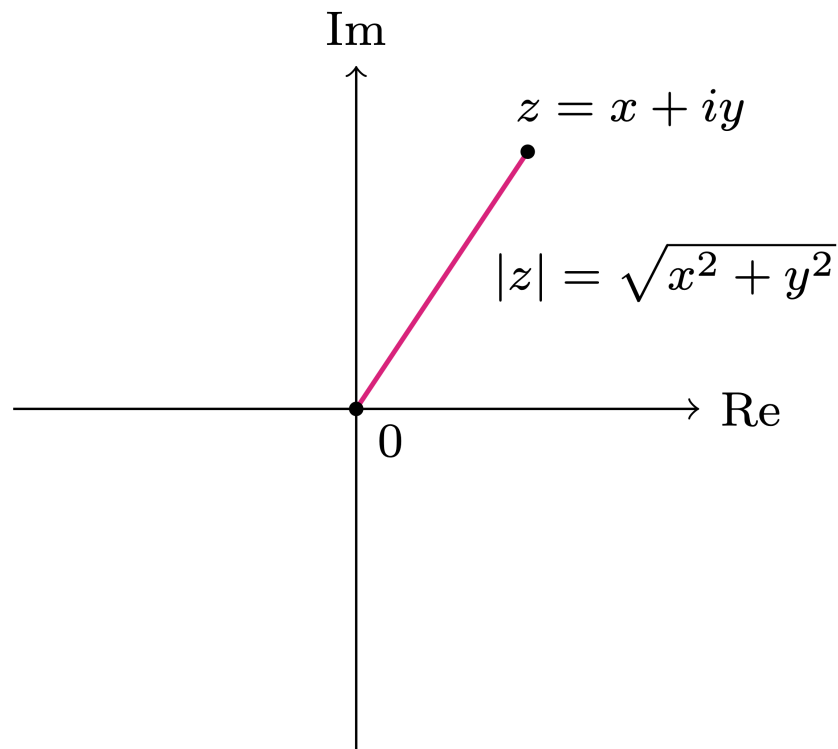
Figure 5.2: A point $z = x + iy \in \mathbb{C}$ can be represented on the complex plane by the point of coordinates (x, y). The distance between z and 0 is given by $|z| = \sqrt{z^2 + y^2}$.

see Figure 5.2. We give a name to this quantity.

Definition 5.21: Modulus

The **modulus** of a complex number $z = x + iy$ is defined by

$$|z| := \sqrt{x^2 + y^2}\,.$$

Note that the distance between z and 0 is always a non-negative number.

Remark 5.22: Modulus of Real numbers

A real number $x \in \mathbb{R}$ can be written as
$$x = x + 0i \in \mathbb{C}\,.$$
Hence the modulus of x is given by
$$|x| = \sqrt{x^2 + 0^2} = \sqrt{x^2}\,.$$
The above coincides with the absolute value of x. This explains why the notation for modulus in $\mathbb{C}$ is the same as the one for absolute value in $\mathbb{R}$.

We can now define the distance between two complex numbers.

Definition 5.23: Distance in $\mathbb{C}$

Given $z_1, z_2 \in \mathbb{C}$, we define the **distance** between z_1 and z_2 as the quantity

$$|z_1 - z_2|\,.$$

The geometric intuition of why the quantity $|z_1 - z_2|$ is defined as the distance between z_1 and z_2 is given in Figure 5.3.

Theorem 5.24

Given $z_1, z_2 \in \mathbb{C}$, we have

$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\,.$$
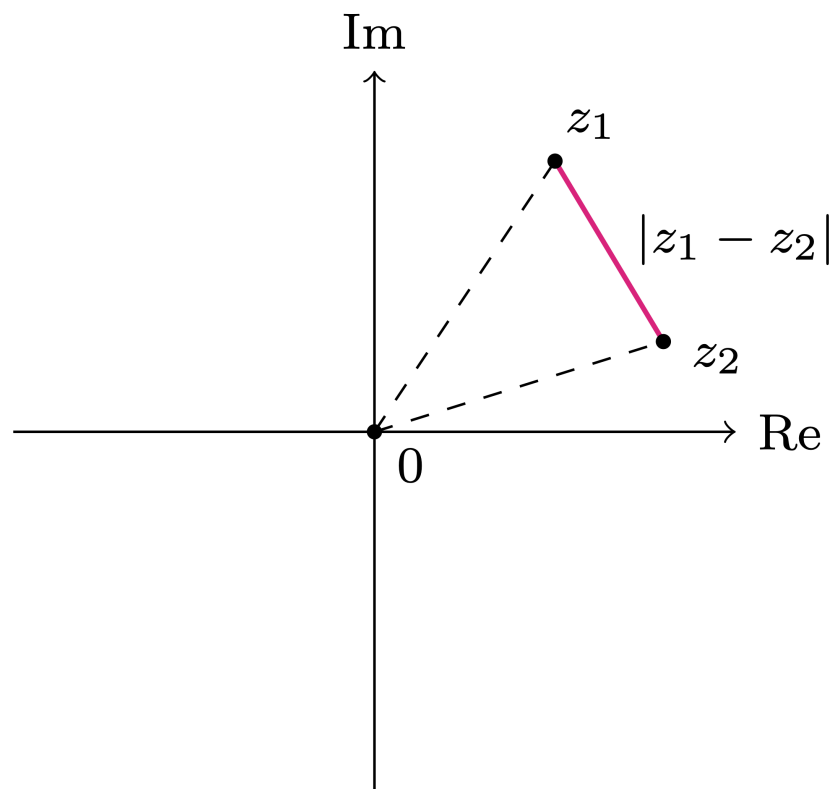
Figure 5.3: The difference $z_1 - z_2$ of the two points $z_1, z_2 \in \mathbb{C}$ is given by the magenta vector. We define $|z_1 - z_2|$ as the distance between z_1 and z_2.

Proof

We have
$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2).$$

Therefore, by definition of modulus,
$$|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Example 5.25

The distance between
$$z = 2 - 4i, \quad w = -5 + i$$

is given by
$$\begin{aligned}
|z - w| &= |(2 - 4i) - (-5 + i)| \\
&= |7 - 5i| \\
&= \sqrt{7^2 + (-5)^2} \\
&= \sqrt{74}
\end{aligned}$$

5.3.2 Properties of modulus

The modulus has the following properties.

Theorem 5.26

Let $z, z_1, z_2 \in \mathbb{C}$. Then

1. $|z_1 \cdot z_2| = |z_1| |z_2|$

2. $|z^n| = |z|^n$ for all $n \in \mathbb{N}$

3. $z \cdot \bar{z} = |z|^2$

Proof

Part 1. We have
$$\begin{aligned}
z_1 \cdot z_2 &= (x_1 + iy_1) \cdot (x_2 + iy_2) \\
&= (x_1 x_2 - y_1 y_2) + i(x_2 y_1 + x_1 y_2)
\end{aligned}$$

and therefore

$$|z_1 \cdot z_2| = \sqrt{(x_1 x_2 - y_1 y_2)^2 + (x_2 y_1 + x_1 y_2)^2}$$
$$= \sqrt{x_1^2 x_2^2 + y_1^2 y_2^2 + x_2^2 y_1^2 + x_1^2 y_2^2} \,.$$

On the other hand,

$$|z_1| = \sqrt{x_1^2 + y_1^2}\,, \quad |z_2| = \sqrt{x_2^2 + y_2^2}$$

so that

$$|z_1||z_2| = \sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}$$
$$= \sqrt{x_1^2 x_2^2 + y_1^2 y_2^2 + x_2^2 y_1^2 + x_1^2 y_2^2}$$

proving that $|z_1 \cdot z_2| = |z_1||z_2|$.

Part 2. Exercise. It easily follows from Point 1 and induction.

Part 3. Let $z = x + iy$ for some $x, y \in \mathbb{R}$. Then,

$$z \cdot \bar{z} = (x + iy)(x - iy)$$
$$= x^2 - (iy)^2$$
$$= x^2 + y^2$$
$$= |z|^2$$

The modulus in $\mathbb{C}$ satisfies the triangle inequality.

Theorem 5.27:   Triangle inequality in $\mathbb{C}$

For all $x, y, z \in \mathbb{C}$,

1. $|x + y| \leq |x| + |y|$

2. $|x - z| \leq |x - y| + |y - z|$

Proof

Part 1. Suppose that $x = a + ib$ and $y = c + id$ for $a, b, c, d \in \mathbb{R}$. Then,

$$|x + y| = |(a + c) + i(b + d)| = \sqrt{(a + c)^2 + (b + d)^2}\,.$$

Therefore the inequality

$$|x + y| \leq |x| + |y| \tag{5.2}$$

is equivalent to

$$\sqrt{(a+c)^2 + (b+d)^2} \leq \sqrt{a^2 + b^2} + \sqrt{c^2 + d^2}. \tag{5.3}$$

Now note that, for $A, B \in \mathbb{R}$, we have that

$$A^2 \leq B^2 \quad \Longrightarrow \quad |A| \leq |B|. \tag{5.4}$$

In the two reverse implications $\Longleftarrow$ below we will use (5.4):

$$\sqrt{(a+c)^2 + (b+d)^2} \leq \sqrt{a^2 + b^2} + \sqrt{c^2 + d^2}$$

$$\Longleftarrow (a+c)^2 + (b+d)^2 \leq \left(\sqrt{a^2 + b^2} + \sqrt{c^2 + d^2}\right)^2$$

$$\Longleftrightarrow a^2 + 2ac + c^2 + b^2 + 2bd + d^2 \leq a^2 + b^2 + 2\sqrt{a^2+b^2}\sqrt{c^2+d^2} + c^2 + d^2$$

$$\Longleftrightarrow ac + bd \leq \sqrt{a^2 + b^2}\sqrt{c^2 + d^2}$$

$$\Longleftarrow (ac + bd)^2 \leq \left(a^2 + b^2\right)\left(c^2 + d^2\right)$$

$$\Longleftrightarrow a^2c^2 + 2abcd + b^2d^2 \leq a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2$$

$$\Longleftrightarrow a^2d^2 + b^2c^2 - 2abcd \geq 0$$

$$\Longleftrightarrow (ad - bc)^2 \geq 0.$$

This last statement is clearly true, since $ad - bc \in \mathbb{R}$. Therefore (5.3) holds, and so (5.2) follows.
Part 2. Using (5.2) we estimate

$$|x - z| = |x - y + y - z| \leq |x - y| + |y - z|.$$

Remark 5.28: Geometric interpretation of triangle inequality

We finally have a justification of why the inequality

$$|x - z| \leq |x - y| + |y - z|$$

is called **triangle inequality**: By drawing three points $x, y, z \in \mathbb{C}$ in the complex plane, the distance between x and z is shorter than the distance to go from x to z via the point y, see Figure 5.4.

5.4 Polar coordinates

We have seen that we can identify a complex number $z = x + iy$ by a point in the complex plane with Cartesian coordinates (x, y). We can also specify the point (x, y) by using the so-called polar coordinates (ρ, θ), where
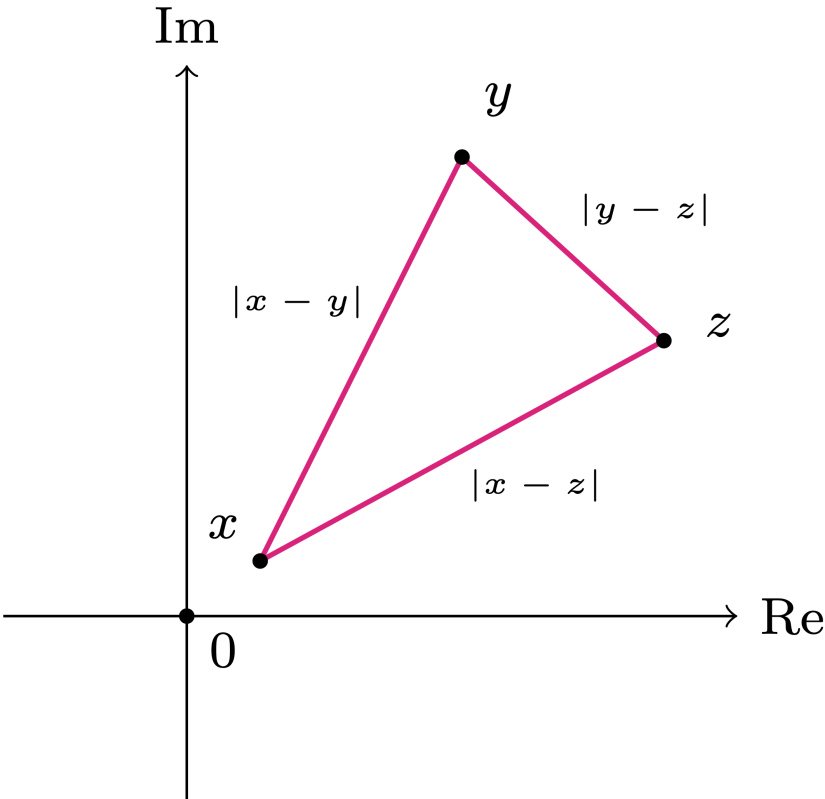
Figure 5.4: Let $x, y, z \in \mathbb{C}$. The distance between x and z is shorter than the distance to go from x to z via the point y.

- ρ is the distance between z and the origin

$$\rho = |z| = \sqrt{x^2 + y^2}$$

- θ is the angle between the line connecting the origin and z and the positive real axis, see Figure 5.5.
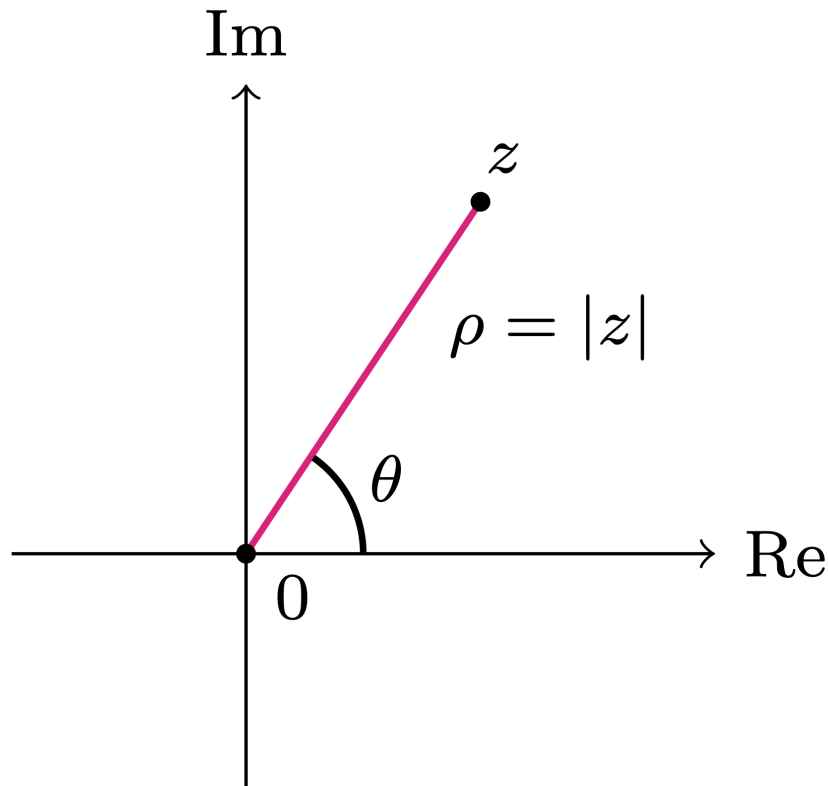
Figure 5.5: Polar coordinates (ρ, θ) for the complex number $z \in \mathbb{C}$.

We give such angle a name.

Definition 5.29: Argument

Let $z \in \mathbb{C}$. The angle θ between the line connecting the origin and z and the positive real axis is called the **argument** of z, and is denoted by

$$\theta := \arg(z).$$

Warning

We always use angles in radians, not degrees. Make sure your calculator is set to radians if you want to use it to compute angles.

Remark 5.30: Principal Value

The argument of a complex number is not uniquely defined. We can always add an integer number of times 2π to the argument to specify the same point. We usually use the convention to choose the argument in the interval $(-\pi, \pi]$. This is called the **principal value** of the argument function. Therefore the complex numbers in the upper half plane have a positive argument, and in the lower half plane have a negative argument.

Example 5.31

We have the following arguments:

$$\arg(1) = 0 \qquad\qquad \arg(i) = \frac{\pi}{2}$$

$$\arg(-1) = \pi \qquad\qquad \arg(-i) = -\frac{\pi}{2}$$

$$\arg(1 + i) = \frac{1}{4}\pi \qquad\qquad \arg(-1 - i) = -\frac{3}{4}\pi$$

We can represent non-zero complex numbers in polar coordinates.

Theorem 5.32: Polar coordinates

Let $z \in \mathbb{C}$ with $z = x + iy$ and $z \neq 0$. Then

$$x = \rho \cos(\theta), \quad y = \rho \sin(\theta),$$

where

$$\rho = \sqrt{x^2 + y^2}, \quad \theta = \arg(z).$$

The proof of Theorem 5.32 is trivial, and is based on basic trigonometry and definition of $\arg(z)$. Complex numbers in polar form can be useful. We give a name to such polar form.

Definition 5.33: Trigonometric form

Let $z \in \mathbb{C}$. The trigonometric form of z is

$$z = |z| \left[\cos(\theta) + i \sin(\theta)\right],$$

where $\theta = \arg(z)$.

Let us make an example.

Example 5.34

Suppose that we have polar coordinates

$$\rho = \sqrt{8}, \quad \theta = \frac{3}{4}\pi$$

We compute

$$x = \rho\cos(\theta) = \sqrt{8}\cos\left(\frac{3}{4}\pi\right) = -\frac{\sqrt{8}\sqrt{2}}{2} = -2$$

$$y = \rho\sin(\theta) = \sqrt{8}\sin\left(\frac{3}{4}\pi\right) = \frac{\sqrt{8}\sqrt{2}}{2} = 2\,.$$

The complex number z corresponding to the polar coordinates (ρ, θ) is

$$z = x + iy = -2 + 2i\,.$$

The trigonometric form of z is

$$z = \sqrt{8}\left[\cos\left(\frac{3}{4}\pi\right) + i\sin\left(\frac{3}{4}\pi\right)\right]\,.$$

As a consequence of Theorem 5.32 we obtain a formula for computing the argument.

Corollary 5.35: Computing $\arg(z)$

Let $z \in \mathbb{C}$ with $z = x + iy$ and $z \neq 0$. Then

$$\arg(z) = \begin{cases} \arctan\left(\dfrac{y}{x}\right) & \text{if } x > 0 \\ \arctan\left(\dfrac{y}{x}\right) + \pi & \text{if } x < 0 \text{ and } y \geq 0 \\ \arctan\left(\dfrac{y}{x}\right) - \pi & \text{if } x < 0 \text{ and } y < 0 \\ \dfrac{\pi}{2} & \text{if } x = 0 \text{ and } y > 0 \\ -\dfrac{\pi}{2} & \text{if } x = 0 \text{ and } y < 0 \end{cases}$$

where arctan is the inverse of tan.

> **Proof**
>
> Using the polar coordinates formulas from Theorem 5.32 we have
>
> $$\frac{y}{x} = \frac{\rho \sin(\theta)}{\rho \cos(\theta)} = \tan(\theta).$$
>
> The thesis can be obtained by carefully inverting the tangent.

> **Example 5.36**
>
> We want to compute the arguments of the complex numbers
>
> $$3 + 4i, \quad 3 - 4i, \quad -3 + 4i, \quad -3 - 4i.$$
>
> Using the formula for arg in Corollary 5.35 we have
>
> $$\arg(3 + 4i) = \arctan\left(\frac{4}{3}\right)$$
> $$\arg(3 - 4i) = \arctan\left(-\frac{4}{3}\right) = -\arctan\left(\frac{4}{3}\right)$$
> $$\arg(-3 + 4i) = \arctan\left(-\frac{4}{3}\right) + \pi = -\arctan\left(\frac{4}{3}\right) + \pi$$
> $$\arg(-3 - 4i) = \arctan\left(\frac{4}{3}\right) - \pi$$

5.5 Exponential form

We have seen that we can represent complex numbers in

- Cartesian form
- Trigonometric form

We now introduce a third way of representing complex numbers: the exponential form. For this, we need Euler's identity:

> **Theorem 5.37:** Euler's identity
>
> For all $\theta \in \mathbb{R}$ it holds
>
> $$e^{i\theta} = \cos(\theta) + i \sin(\theta).$$

Proof

The proof of this theorem uses Taylor power series. Note that we have not introduced what series are, yet, so we just assume that everything below makes sense and actually exists. We have the following Taylor series at $x_0 = 0$ that you might know from calculus:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \dots$$

$$\sin(x) = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

The above identities also holds for $x \in \mathbb{C}$. Hence we can substitute $x = i\theta$ in the series for e^x to obtain

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} \dots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - i\frac{\theta^7}{7!} + \dots$$

$$= \cos(\theta) + i\sin(\theta),$$

where we used that $i^2 = -1$ in the second equality, and the third equality follows by observing that all terms with an even power of θ are exactly the terms in the expansion of $\cos(\theta)$ and all terms with an odd power of θ are exactly the terms in the expansion of $\sin(\theta)$ multiplied by i.

Theorem 5.38

For all $\theta \in \mathbb{R}$ it holds

$$\left|e^{i\theta}\right| = 1 .$$

Proof

From Euler's identity in Theorem 5.37 we get

$$\left|e^{i\theta}\right| = |\cos(\theta) + i\sin(\theta)| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = 1 .$$

Theorem 5.39

Let $z \in \mathbb{C}$ with $z = x + iy$ and $z \neq 0$. Then

$$z = \rho e^{i\theta} ,$$

where

$$\rho = \sqrt{x^2 + y^2} = |z|, \quad \theta = \arg(z).$$

Proof

By Theorem 5.32 we have

$$x = \rho \cos(\theta), \quad y = \rho \sin(\theta).$$

Hence

$$\begin{aligned} z &= x + iy \\ &= \rho \cos(\theta) + i\rho \sin(\theta) \\ &= \rho e^{i\theta}, \end{aligned}$$

where in the last line we used Euler's identity in Theorem 5.37.

Definition 5.40: Exponential form

A complex number $z \in \mathbb{C}$ is in **exponential form** if

$$z = \rho e^{i\theta} = |z| \, e^{i \arg(z)}.$$

Example 5.41

From Example 5.34 we know that

$$z = -2 + 2i$$

can be written in trigonometric form as

$$z = \sqrt{8} \left[\cos\left(\frac{3}{4}\pi\right) + i \sin\left(\frac{3}{4}\pi\right) \right].$$

By Euler's identity we hence obtain the exponential form

$$z = \sqrt{8} e^{i\frac{3}{4}\pi}.$$

Remark 5.42: Periodicity of exponential

For all $k \in \mathbb{Z}$ we have

$$e^{i\theta} = e^{i(\theta + 2\pi k)} . \tag{5.5}$$

As we did for the principal value of the argument, also for the exponential form we select $\theta \in (-\pi, \pi]$. In particular equation (5.5) is saying that the complex exponential is 2π-periodic.

Equation (5.5) follows immediately by Euler's identity and periodicity of cos and sin, since

$$e^{i(\theta + 2\pi k)} = \cos(\theta + 2\pi k) + i \sin(\theta + 2\pi k)$$
$$= \cos(\theta) + i \sin(\theta) = e^{i\theta} .$$

The exponential for is very useful for computing products and powers of complex numbers.

Proposition 5.43

Let $z, z_1, z_2 \in \mathbb{C}$ and suppose that

$$z = \rho e^{i\theta} , \quad z_1 = \rho_1 e^{i\theta_1} , \quad z_2 = \rho_2 e^{i\theta_2} .$$

We have

$$z_1 \cdot z_2 = \rho_1 \rho_2 e^{i(\theta_1 + \theta_2)} , \quad z^n = \rho^n e^{in\theta} ,$$

for all $n \in \mathbb{N}$.

The proof follows immediately by the properties of the exponential. Let us see some applications of Propostion 5.43.

Example 5.44

Suppose we want to compute $(-2 + 2i)^4$. We could do this by means of the binomial theorem:

$$(-2 + 2i)^4 = (-2)^4 + \binom{4}{1}(-2)^3 \cdot 2i + \binom{4}{2}(-2)^2 \cdot (2i)^2 + \binom{4}{3}(-2) \cdot (2i)^3 + (2i)^4$$
$$= 16 - 4 \cdot 8 \cdot 2i - 6 \cdot 4 \cdot 4 + 4 \cdot 2 \cdot 8i + 16$$
$$= 16 - 64i - 96 + 64i + 16 = -64 .$$

Using the exponential form simplifies this calculation. Indeed, we know that

$$-2 + 2i = \sqrt{8} e^{i\frac{3}{4}\pi}$$

by Example 5.41. Hence

$$(-2 + 2i)^4 = \left(\sqrt{8} e^{i\frac{3}{4}\pi} \right)^4 = 8^2 e^{i3\pi} = -64 ,$$

where we used that

$$e^{i3\pi} = e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1$$

by 2π periodicity of e^z and Euler's identity.

Example 5.45

Suppose we want to compute

$$i^i.$$

It is not even clear how to do this calculation in Cartesian form. However, we know that

$$|i| = 1, \quad \arg(i) = \frac{\pi}{2}.$$

Hence we can write i in exponential form

$$i = |i|e^{i\arg(i)} = e^{i\frac{\pi}{2}}.$$

Therefore

$$i^i = \left(e^{i\frac{\pi}{2}}\right)^i = e^{i^2\frac{\pi}{2}} = e^{-\frac{\pi}{2}}.$$

5.6 Fundamental Theorem of Algebra

We started the introduction to complex numbers with the following question:

Question 5.46

Is there a number $x \in \mathbb{R}$ such that

$$x^2 = -1 \ ? \tag{5.6}$$

The answer is no. For this reason we introduced the complex number i, which satisfies

$$i^2 = -1.$$

Therefore (5.6) has solution in $\mathbb{C}$, with $x = i$. We also have that

$$(-i)^2 = (-1)^2 i^2 = -1.$$

Hence (5.6) has two solutions in $\mathbb{C}$, given by

$$x_1 = i, \quad x_2 = -i.$$

It turns out that the set $\mathbb{C}$ is so large that we are not only able to solve (5.6), but in fact any polynomial equation.

Theorem 5.47:   Fundamental theorem of algebra

Let $p_n(x)$ be a polynomial of degree n with complex coefficients, i.e.,

$$p_n(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0,$$

for some coefficients $a_n, \ldots, a_0 \in \mathbb{C}$ with $a_n \neq 0$. Then there exist $z_1, \ldots, z_n \in \mathbb{C}$ such that

$$p_n(x) = a_n (x - z_1)(x - z_2) \cdots (x - z_n) . \tag{5.7}$$

Hence, the polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0 \tag{5.8}$$

has solutions $z_1, \ldots, z_n$.

Theorem 5.47 says that every polynomial of degree n has n zeros, sometimes also called roots, i.e., n solutions to (5.8). We call the expression (5.7) a factorization of the polynomial p_n.

Several proofs of Theorem 5.47 exist in the literature, but they all use mathematical tools which are out of reach for now. Therefore we will not show a proof. For example one can prove Theorem 5.47 by

- Liouville's theorem (complex analysis)
- Homotopy arguments (general topology)
- Fundamental Theorem of Galois Theory (algebra)

Example 5.48

The equation

$$x^2 = -1 \tag{5.9}$$

is equivalent to

$$p(x) = 0, \quad p(x) := x^2 + 1 .$$

Since p has degree $n = 2$, the Fundamental Theorem of Algebra tells us that there are two solutions to (5.9). We have already seen that these two solutions are $x = i$ and $x = -i$. Then

$$p(x) = x^2 + 1 = (x - i)(x + i) .$$

Example 5.49

Suppose we want to solve

$$x^4 - 1 = 0 \,. \qquad (5.10)$$

The associated polynomial equation is

$$p(x) = 0 \,, \quad p(x) := x^4 - 1 \,.$$

Since p has degree $n = 4$, the Fundamental Theorem of Algebra tells us that there are 4 solutions to (5.10). Let us find such solutions. We use the well known formula

$$a^2 - b^2 = (a + b)(a - b) \,, \quad \forall \, a, b \in \mathbb{R} \,,$$

to factorize p. We get:

$$p(x) = (x^4 - 1) = (x^2 + 1)(x^2 - 1) \,.$$

We know that

$$x^2 + 1 = 0$$

has solutions $x = \pm i$. Instead

$$x^2 - 1 = 0$$

has solutions $x = \pm 1$. Hence, the four solutions of (5.10) are given by $x = 1, -1, i, -i$ and

$$p(x) = x^4 - 1 = (x - 1)(x + 1)(x - i)(x + i) \,.$$

Definition 5.50

Suppose that the polynomial p_n factorizes as

$$p_n(x) = a_n(x - z_1)^{k_1}(x - z_2)^{k_2} \cdots (x - z_m)^{k_m}$$

with $a_n \neq 0$, $z_1, \dots, z_m \in \mathbb{C}$ and $k_1, \dots, k_m \in \mathbb{N}$, $k_i \geq 1$. In this case p_n has degree

$$n = k_1 + \dots + k_m = \sum_{i=1}^{m} k_i \,.$$

We have that z_i is solves the equation

$$p_n(x) = 0$$

exactly k_i times. We call k_i the **multiplicity** of the solution z_i.

> **Example 5.51**
>
> The equation
> $$(x-1)(x-2)^2(x+i)^3 = 0$$
> has 6 solutions:
>
> - $x = 1$ with multiplicity 1
> - $x = 2$ with multiplicity 2
> - $x = -i$ with multiplicity 3

5.7 Solving polynomial equations

The non-factorized version of the polynomial
$$p(x) = (x-1)(x-2)^2(x+i)^3$$
from Example 5.51 is
$$
\begin{aligned}
p(x) = & x^6 - (5-3i)x^5 + (5-15i)x^4 \\
& + (11+23i)x^3 - (24+7i)x^2 + (12-8i)x + 4i
\end{aligned}
$$
We therefore have the following natural question.

> **Question 5.52**
>
> The Fundamental Theorem of Algebra states that
> $$p_n(x) = 0 \tag{5.11}$$
> has n complex solutions. How do we find such solutions?

The answer is that there is no general way to solve (5.11) when $n \geq 5$. This is the content of the Abel-Ruffini Theorem.

> **Theorem 5.53:** Abel-Ruffini
>
> There is no elementary solution formula to the polynomial equation
> $$p_n(x) = 0$$
> with p_n polynomial of degree n, with $n \geq 5$.

Similarly to the Fundamental Theorem of Algebra, the proof of the Abel-Ruffini Theorem is out of reach for our current mathematical knowledge. A proof can be carried out, for example, using Galois Theory.

There are however explicit formulas for solving (5.11) when p_n has degree $n = 2, 3, 4$. For $n = 2$ we can use the well-known quadratic formula.

Proposition 5.54: Quadratic formula

Let $a, b, c \in \mathbb{R}, a \neq 0$ and consider the equation

$$ax^2 + bx + c = 0.$$ (5.12)

Define

$$\Delta := b^2 - 4ac.$$

The following hold:

- If $\Delta > 0$ then (5.12) has two distinct real solutions given by

$$x_1 = \frac{-b - \sqrt{\Delta}}{2a}, \quad x_2 = \frac{-b + \sqrt{\Delta}}{2a}.$$

- If $\Delta = 0$ then (5.12) has one solution with multiplicity 2. Such solution is given by

$$x_1 = \frac{-b}{2a}.$$

- If $\Delta < 0$ then (5.12) has two distinct complex solutions given by

$$x_1 = \frac{-b - i\sqrt{-\Delta}}{2a}, \quad x_2 = \frac{-b + i\sqrt{-\Delta}}{2a},$$

where $\sqrt{-\Delta}$ is a real number, since $-\Delta > 0$.

Moreover, if $\Delta \neq 0$ we have

$$ax^2 + bx + c = a(x - x_1)(x - x_2),$$

while if $\Delta = 0$ then

$$ax^2 + bx + c = a(x - x_1)^2.$$

Example 5.55

Suppose we want to solve

$$3x^2 - 6x + 2 = 0.$$

We have that

$$\Delta = (-6)^2 - 4 \cdot 3 \cdot 2 = 12 > 0$$

Therefore the equation has two distinct real solutions, given by

$$x = \frac{-(-6) \pm \sqrt{12}}{2 \cdot 3} = \frac{6 \pm \sqrt{12}}{6} = 1 \pm \frac{\sqrt{3}}{3}$$

In particular we have the factorization

$$3x^2 - 6x + 2 = 3\left(x - 1 - \frac{\sqrt{3}}{3}\right)\left(x - 1 + \frac{\sqrt{3}}{3}\right).$$

Example 5.56

Suppose we want to solve

$$4x^2 - 8x + 4 = 0.$$

We have that

$$\Delta = (-8)^2 - 4 \cdot 4 \cdot 4 = 0.$$

Therefore there exists one solution with multiplicity 2. This is given by

$$x = \frac{-(-8)}{2 \cdot 4} = 1.$$

In particular we have the factorization

$$4x^2 - 8x + 4 = 4(x - 1)^2.$$

Example 5.57

Consider

$$x^2 + 2x + 3 = 0.$$

We have

$$\Delta = 2^2 - 4 \cdot 1 \cdot 3 = -8 < 0.$$

Therefore there are two complex solutions given by

$$x = \frac{-2 \pm i\sqrt{8}}{2 \cdot 1} = -1 \pm i\sqrt{2}.$$

In particular we have the factorization

$$x^2 + 2x + 3 = (x + 1 - i\sqrt{2})(x + 1 + i\sqrt{2}).$$

So far we have considered the polynomial equation

$$ax^2 + bx + c = 0,$$

for $a, b, c \in \mathbb{R}$ and $a \neq 0$.

Question 5.58

What if $a, b, c \in \mathbb{C}$?

If $a, b, c \in \mathbb{C}$ then we might have

$$\Delta := b^2 - 4ac \in \mathbb{C}.$$

Therefore it is not clear how to compute

$$\sqrt{\Delta}.$$

However, we can still use the quadratic equation.

Proposition 5.59: Generalization of quadratci formula

Let $a, b, c \in \mathbb{C}, a \neq 0$. The two solutions to

$$ax^2 + bx + c = 0$$

are given by

$$x_1 = \frac{-b + S_1}{2a}, \quad x_2 = \frac{-b + S_2}{2a},$$

where S_1 and S_2 are the two solutions to

$$z^2 = \Delta, \quad \Delta := b^2 - 4ac.$$

Remark 5.60

Suppose that $\Delta \in \mathbb{R}$. The equation

$$z^2 = \Delta$$

has the following solutions:

- If $\Delta > 0$ there are two real solutions

$$S_1 = -\sqrt{\Delta}, \quad S_2 = \sqrt{\Delta}$$

- If $\Delta = 0$ then 0 is the only solution with multiplicity 2. Hence

$$S_1 = S_2 = 0$$

- If $\Delta < 0$ there are two complex solutions

$$S_1 = -i\sqrt{-\Delta}, \quad S_2 = i\sqrt{-\Delta}$$

Therefore the solutions

$$x_1 = \frac{-b + S_1}{2a}, \quad x_2 = \frac{-b + S_2}{2a},$$

given in Proposition 5.54 coincide with the ones given in Proposition 5.59.

Example 5.61

Let us see an application of Proposition 5.59. Consider the equation

$$\frac{1}{2}x^2 - (3 + i)x + (4 - i) = 0. \tag{5.13}$$

We have

$$\begin{aligned}
\Delta &= (-(3 + i))^2 - 4 \cdot \frac{1}{2} \cdot (4 - i) \\
&= 8 + 6i - 8 + 2i \\
&= 8i.
\end{aligned}$$

Therefore $\Delta \in \mathbb{C}$. We have to find solutions S_1 and S_2 to the equation

$$z^2 = \Delta = 8i. \tag{5.14}$$

We look for solutions of the form $z = x + iy$. Then we must have that

$$z^2 = (a + ib)^2 = a^2 - b^2 + 2abi = 8i.$$

Thus

$$a^2 - b^2 = 0, \quad 2ab = 8.$$

From the first equation we conclude that $|a| = |b|$. From the second equation we have that $ab = 4$, and therefore a and b must have the same sign. Hence $a = b$, and

$$2ab = 8 \quad \implies \quad a = b = \pm 2.$$

From this we conclude that the solutions to (5.14) are

$$S_1 = 2 + 2i, \quad S_2 = -2 - 2i.$$

Hence the solutions to (5.13) are

$$\begin{aligned}
x_1 &= \frac{3 + i + S_1}{2 \cdot \frac{1}{2}} = 3 + i + S_1 \\
&= 3 + i + 2 + 2i = 5 + 3i,
\end{aligned}$$

and

$$x_2 = \frac{3 + i + S_2}{2 \cdot \frac{1}{2}} = 3 + i + S_2$$

$$= 3 + i - 2 - 2i = 1 - i \,.$$

In the above example it was a bit laborious to compute S_1 and S_2. In the next section we will see an easier way to solve problems of the form $z^2 = \Delta$.

Remark 5.62: Polynomial equations of order $n = 3, 4$

For polynomial equations
$$p_n(x) = 0$$
with p_n of degree $n = 3, 4$ similar methods exist. However the solution formulas for such equations are really complicated, and we do not cover them here.
Still, it is sometimes possible to solve equations of degree higher than 2, in case it is obvious from inspection that a certain number is a solution, e.g., when $x = -1, 0, 1$ is a solution.

Example 5.63

Consider the equation
$$x^3 - 7x^2 + 6x = 0 \,.$$

It is clear that $x = 0$ is a solution and that we can write

$$x^3 - 7x^2 + 6x = x \left(x^2 - 7x + 6 \right) \,.$$

We could now use the quadratic formula to find the remaining two roots, but we can also directly observe that also $x = 1$ is a solution, so that $x - 1$ divides $x^2 - 7x + 6$. Using polynomial long division, we find that
$$\frac{x^2 - 7x + 6}{x - 1} = x - 6 \,,$$
see Figure 5.6. Therefore the last solution is $x = 6$, and

$$x^3 - 7x^2 + 6x = x(x - 1)(x - 6) \,.$$

$$
\begin{array}{r}
x - 6 \\
x - 1 \overline{)\ x^2 - 7x + 6\ } \\
\underline{-\ x^2\ + x} \\
-6x + 6 \\
\underline{6x - 6} \\
0
\end{array}
$$

Figure 5.6: Polynomial long division between $x^2 - 7x + 6$ and $x - 1$.

$$
\begin{array}{r}
2x + 3 \\
3x^2 - 2x - 1 \overline{)\ 6x^3 + 5x^2\qquad - 7\ } \\
\underline{-\ 6x^3 + 4x^2 + 2x} \\
9x^2 + 2x - 7 \\
\underline{-\ 9x^2 + 6x + 3} \\
8x - 4
\end{array}
$$

Figure 5.7: Example of polynomial long division between $6x^3 + 5x^2 - 7$ and $3x^2 - 2x - 1$.

Example 5.64

Suppose we want to solve
$$x^3 - 7x + 6 = 0 \, .$$

It is easy to see $x = 1$ is a solution. This means that $x - 1$ divides $x^3 - 7x + 6$, so we can compute by using polynomial long division,

$$\frac{x^3 - 7x + 6}{x - 1} = x^2 + x - 6 \, ,$$

see Figure 5.8. For the remaining two solutions, we can use the quadratic formula to obtain that also $x = 2$ and $x = -3$ are solutions. Thus

$$x^3 - 7x + 6 = (x - 1)(x - 2)(x + 3) \, .$$

$$
\begin{array}{r}
x^2 + x - 6 \\
x - 1 \overline{) \, x^3 \qquad\quad - 7x + 6} \\
\underline{-x^3 + x^2} \qquad\qquad\quad \\
x^2 - 7x \qquad \\
\underline{-x^2 + x} \qquad \\
-6x + 6 \\
\underline{6x - 6} \\
0
\end{array}
$$

Figure 5.8: Polynomial long division between $x^3 - 7x + 6$ and $x - 1$.

5.8 Roots of unity

Problem

Let $n \in \mathbb{N}$. We want to find all complex solutions to

$$z^n = 1 \, . \tag{5.15}$$

Note that $z = 1$ is always a solution to (5.15) if n is even. In such case also $z = -1$ is a solution. If we were only looking for solutions in $\mathbb{R}$, these two would be the only solutions.

However, the Fundamental Theorem of Algebra, see Theorem 5.47, tells us that there are n complex solutions to (5.15).

Question 5.65

Is there a way to find all n solutions?

Example 5.66

We have seen in Example 5.49 that the solutions to

$$x^4 = 1$$

are $x = -1, 1, i, -i$. However we deduced this with a procedure which does not seem to generalize well to other exponents.

The trick to find all n solutions to (5.15) is to use the exponential form.

Theorem 5.67

Let $n \in \mathbb{N}$ and consider the equation

$$z^n = 1 \, . \tag{5.16}$$

All the n solutions to (5.16) are given by

$$z_k = \exp\left(i\frac{2\pi k}{n}\right), \quad k = 0, \dots, n-1 \, ,$$

where $\exp(x)$ denotes e^x.

Proof

Rewrite 1 in exponential form:
$$1 = |1|e^{i\arg(1)} = e^{i2\pi k}, \quad k \in \mathbb{Z}.$$

Therefore (5.16) is equivalent to
$$z^n = e^{i2\pi k}.$$

By the properties of the exponential, we see that the above is solved by
$$z_k = \exp\left(i\frac{2\pi k}{n}\right), \quad k \in \mathbb{Z}.$$

By choosing $k = 0, \dots, n-1$ we obtain n different solutions.

Definition 5.68

The solutions to
$$z^n = 1$$
are called the **roots of unity**.

Example 5.69

The solutions to
$$z^4 = 1$$
are given by
$$z_k = \exp\left(i\frac{2\pi k}{4}\right) = \exp\left(i\frac{\pi k}{2}\right).$$

By taking $k = 0, 1, 2, 3$, we obtain the four solutions

$$z_0 = e^{i0} = 1, \qquad\qquad z_1 = e^{i\frac{\pi}{2}} = i,$$
$$z_2 = e^{i\pi} = -1, \qquad\qquad z_3 = e^{i\frac{3\pi}{2}} = -i.$$

Note that for $k = 4$ we would again get the solution $z = e^{i2\pi} = 1$.

Example 5.70

The solutions to
$$z^3 = 1$$

are given by

$$z_k = \exp\left(i\frac{2\pi k}{3}\right).$$

By taking $k = 0, 1, 2$, we obtain the three solutions

$$z_0 = e^{i0} = 1, \quad z_1 = e^{i\frac{2\pi}{3}}, \quad z_2 = e^{i\frac{4\pi}{3}}.$$

We can also convert z_1 and z_2 to trigonometric form:

$$z_1 = e^{i\frac{2\pi}{3}} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

and

$$z_2 = e^{i\frac{4\pi}{3}} = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

5.9 Roots in $\mathbb{C}$

Problem

Let $n \in \mathbb{N}$ and $c \in \mathbb{C}$. We want to find the n-th roots of c. This means we want to find all complex solutions to

$$z^n = c.$$

The Fundamental Theorem of Algebra ensures that the above has n complex solutions. To find these solutions, we pass to the exponential form.

Theorem 5.71

Let $n \in \mathbb{N}$, $c \in \mathbb{C}$ and consider the equation

$$z^n = c. \tag{5.17}$$

All the n solutions to (5.17) are given by

$$z_k = \sqrt[n]{|c|}\,\exp\left(i\,\frac{\theta + 2\pi k}{n}\right), \quad k = 0, \ldots, n-1,$$

where $\sqrt[n]{|c|}$ is the n-th root of the real number $|c|$, and $\theta = \arg(c)$.

Proof

Write c in exponential form:
$$c = |c|e^{i\theta} = |c|e^{i(\theta + 2\pi k)}, \quad k \in \mathbb{Z},$$

where $\theta = \arg(c)$. Therefore (5.17) is equivalent to

$$z^n = |c|e^{i(\theta + 2\pi k)}.$$

By the properties of the exponential, we see that the above is solved by

$$z_k = \sqrt[n]{|c|} \exp\left(i\frac{\theta + 2\pi k}{n}\right), \quad k \in \mathbb{Z}.$$

By choosing $k = 0, \ldots, n-1$ we obtain n different solutions.

Example 5.72

We want to find all the $z \in \mathbb{C}$ such that
$$z^5 = -32.$$

Let $c = -32$. We have
$$|c| = |-32| = 32 = 2^5, \quad \theta = \arg(-32) = \pi.$$

Hence, the solutions are given by

$$z_k = \left(2^5\right)^{\frac{1}{5}} \exp\left(i\pi\frac{1 + 2k}{5}\right), \quad k \in \mathbb{Z}.$$

By taking $k = 0, 1, 2, 3, 4$ we get the solutions

$$z_0 = 2e^{i\frac{\pi}{5}} \qquad\qquad z_1 = 2e^{i\frac{3\pi}{5}}$$

$$z_2 = 2e^{i\pi} = -2 \qquad\qquad z_3 = 2e^{i\frac{7\pi}{5}}$$

$$z_4 = 2e^{i\frac{9\pi}{5}}$$

Example 5.73

We want to find all the $z \in \mathbb{C}$ such that

$$z^4 = 9\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right).$$

Set

$$c := 9\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right).$$

The complex number c is already in the trigonometric form, so that we can immediately obtain

$$|c| = 9, \quad \theta = \arg(c) = \frac{\pi}{3}.$$

Hence the solutions are given by

$$z_k = \sqrt[4]{9} \, \exp\left(i \, \frac{\pi/3 + 2\pi k}{4}\right)$$
$$= \sqrt{3} \, \exp\left(i\pi \, \frac{1 + 6k}{12}\right)$$

for $k \in \mathbb{Z}$. Choosing $k = 0, 1, 2, 3$ gives the 4 solutions

$$z_0 = \sqrt{3} e^{i\pi \frac{1}{12}} \qquad\qquad z_1 = \sqrt{3} e^{i\pi \frac{7}{12}}$$
$$z_2 = \sqrt{3} e^{i\pi \frac{13}{12}} \qquad\qquad z_3 = \sqrt{3} e^{i\pi \frac{19}{12}}$$

6 Sequences in $\mathbb{R}$

A sequence is an infinite list of real numbers. For example, the following are sequences:

- $(1, 2, 3, 4, \ldots)$
- $(-1, 1, -1, 1, \ldots)$
- $\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right)$

Remark 6.1

- The order of elements in a sequence matters.

  For example
  $$(1, 2, 3, 4, 5, 6, \ldots) \neq (2, 1, 4, 3, 6, 5, \ldots)$$

- A sequence is not a set.

  For example
  $$\{-1, 1, -1, 1, -1, 1, \ldots\} = \{-1, 1\}$$
  but we cannot make a similar statement for the sequence
  $$(-1, 1, -1, 1, -1, 1, \ldots).$$

- The above notation is ambiguous.

  For example the sequence
  $$(1, 2, 3, 4, \ldots)$$
  can continue as
  $$(1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, \ldots).$$

- In the sequence
  $$\left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right)$$
  the elements get smaller and smaller, and closer and closer to 0. We say that this sequence converges to 0, or has 0 as a limit.

We would like to make the notions of sequence and convergence more precise.

6.1 Definition of sequence

We start with the definition of sequence of Real numbers.

Definition 6.2:   Sequence of Real numbers

A sequence a in $\mathbb{R}$ is a function
$$a : \mathbb{N} \to \mathbb{R}.$$

For $n \in \mathbb{N}$, we denote the n-th element of the sequence a by
$$a_n = a(n)$$

and write the sequence as
$$(a_n)_{n\in\mathbb{N}} \ .$$

Notation 6.3

We will sometimes omit the subscript $n \in \mathbb{N}$ and simply write
$$(a_n) \ .$$

In certain situations, we will also write
$$(a_n)_{n=1}^{\infty} \ .$$

Example 6.4

- In general $(a_n)_{n\in\mathbb{N}}$ is the sequence
$$(a_1, a_2, a_3, \dots) \ .$$

- Consider the function
$$a : \mathbb{N} \to \mathbb{N}, \quad n \mapsto 2n .$$
This is also a sequence of real numbers. It can be written as
$$(2n)_{n\in\mathbb{N}}$$
and it represents the sequence of even numbers
$$(2, 4, 6, 8, 10, \dots) .$$

- Let
$$a_n = (-1)^n$$
Then (a_n) is the sequence
$$(-1, 1, -1, 1, -1, 1, \dots) .$$

- $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ is the sequence

$$\left(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \dots\right).$$

6.2 Convergent sequences

We have notice that the sequence $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$ gets close to 0 as n gets large. We would like to say that a_n converges to 0 as n tends to infinity.

To make this precise, we first have to say what it means for two numbers to be *close*. For this we use the notion of absolute value, and say that:

- x and y are close if $|x - y|$ is small.
- $|x - y|$ is called the distance between x and y
- For x to be close to 0, we need that $|x - 0| = |x|$ is small.

Saying that $|x|$ is *small* is not very precise. Let us now give the formal definition of convergent sequence.

Definition 6.5: Convergent sequence

We say that a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ **converges** to $a \in \mathbb{R}$, or equivalently has limit a, denoted by

$$\lim_{n\to\infty} a_n = a$$

if for all $\varepsilon \in \mathbb{R}, \varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}, n \geq N$ it holds that

$$|a_n - a| < \varepsilon.$$

Using quantifiers, we can write this as

$$\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N}\ \text{s.t.}\ \forall\, n \geq N\,,\ |a_n - a| < \varepsilon.$$

If there exists $a \in \mathbb{R}$ such that $\lim_{n\to\infty} a_n = a$, then we say that the sequence $(a_n)_{n\in\mathbb{N}}$ is **convergent**.

Notation 6.6

We will often write

$$a_n \to a$$

---

in place of

$$\lim_{n \to \infty} a_n = a \,.$$

Remark 6.7

- Informally, Definition 6.5 says that, no matter how small we choose ε (as long as it is strictly positive), we always have that a_n has a distance to a of less than or equal to ε from a certain point onwards (i.e., from N onward). The sequence (a_n) may fluctuate wildly in the beginning, but from N onward it should stay within a distance of ε of a.

- In general N depends on ε. If ε is chosen smaller, we might have to take N larger: this means we need to wait longer before the sequence stays within a distance ε from a.

We now prove that the sequence

$$a_n = \frac{1}{n}$$

converges to 0, according to Definition 6.5.

Theorem 6.8

The sequence $\left(\frac{1}{n}\right)_{n \in \mathbb{N}}$ converges to 0 , i.e.,

$$\lim_{n \to \infty} \frac{1}{n} = 0 \,.$$

We give two proofs of the above theorem:

- Long proof, with all the details.
- Short proof, with less details, but still acceptable.

Proof: Proof of Theorem 6.8 (Long version)

We have to show that

$$\lim_{n \to \infty} \frac{1}{n} = 0,$$

which by definition is equivalent to showing that

$$\forall \, \varepsilon > 0 \,, \; \exists \, N \in \mathbb{N} \; \text{ s.t. } \; \forall n \geq N \,, \; \left| \frac{1}{n} - 0 \right| < \varepsilon \,. \tag{6.1}$$

Let $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$N > \frac{1}{\varepsilon}.$$

Such natural number N exists thanks to the Archimedean property. The above implies

$$\frac{1}{N} < \varepsilon, \tag{6.2}$$

Let $n \in \mathbb{N}$ with $n \geq N$. By (6.2) we have

$$\frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

From this we deduce

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$$

Since $n \in \mathbb{N}, n \geq N$ was arbitrary, we have proven that

$$\left| \frac{1}{n} - 0 \right| < \varepsilon, \quad \forall n \geq N. \tag{6.3}$$

Condition (6.3) holds for all $\varepsilon > 0$, for the choice of $N \in \mathbb{N}$ such that

$$N > \frac{1}{\varepsilon}.$$

We have hence shown (6.1), and the proof is concluded.

As the above proof is quite long and includes lots of details, it is acceptable to shorten it. For example:

- We skip some intermediate steps.
- We do not mention the Archimedean property.
- We leave out the conclusion when it is obvious that the statement has been proven.

Proof: Proof of Theorem 6.8 (Short version)

We have to show that

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N, \left| \frac{1}{n} - 0 \right| < \varepsilon.$$

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$N > \frac{1}{\varepsilon}.$$

Let $n \geq N$. Then

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

In Theorem 6.8 we showed that

$$\lim_{n\to\infty} \frac{1}{n} = 0 \,.$$

We can generalise this statement to prove that

$$\lim_{n\to\infty} \frac{1}{n^p} = 0$$

for any $p > 0$ fixed.

Theorem 6.9

For all $p > 0$, the sequence $\left(\frac{1}{n^p}\right)_{n\in\mathbb{N}}$ converges to 0, i.e.,

$$\lim_{n\to\infty} \frac{1}{n^p} = 0$$

Proof

Let $p > 0$. We have to show that

$$\forall \varepsilon > 0 \,, \ \exists N \in \mathbb{N} \ \text{s.t.} \ \forall n \geq N \,, \ \left|\frac{1}{n^p} - 0\right| < \varepsilon \,.$$

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$N > \frac{1}{\varepsilon^{1/p}} \,. \tag{6.4}$$

Let $n \geq N$. Since $p > 0$, we have $n^p \geq N^p$, which implies

$$\frac{1}{n^p} \leq \frac{1}{N^p} \,.$$

By (6.4) we deduce

$$\frac{1}{N^p} < \varepsilon \,.$$

Then

$$\left|\frac{1}{n^p} - 0\right| = \frac{1}{n^p} \leq \frac{1}{N^p} < \varepsilon \,.$$

Question 6.10

Why did we choose $N \in \mathbb{N}$ such that

$$N > \frac{1}{\varepsilon^p}$$

in the above proof?

The answer is: because it works. Finding a number N that makes the proof work requires some *rough work*: Specifically, such rough work consists in finding $N \in \mathbb{N}$ such that the inequality

$$|a_N - a| < \varepsilon$$

is satisfied.

Important

Any *rough work* required to prove convergence must be shown before the formal proof (in assignments).

Example 6.11

Prove that, as $n \to \infty$,

$$\frac{n}{2n + 3} \to \frac{1}{2} .$$

Part 1. Rough Work.

Let $\varepsilon > 0$. We want to find $N \in \mathbb{N}$ such that

$$\left| \frac{n}{2n + 3} - \frac{1}{2} \right| < \varepsilon , \quad \forall\, n \geq N .$$

To this end, we compute:

$$\left| \frac{n}{2n + 3} - \frac{1}{2} \right| = \left| \frac{2n - (2n + 3)}{2(2n + 3)} \right|$$

$$= \left| \frac{-3}{4n + 6} \right|$$

$$= \frac{3}{4n + 6} .$$

Therefore

$$\left| \frac{n}{2n + 3} - \frac{1}{2} \right| < \varepsilon \quad \Longleftrightarrow \quad \frac{3}{4n + 6} < \varepsilon$$

$$\Longleftrightarrow \quad \frac{4n + 6}{3} > \frac{1}{\varepsilon}$$

$$\Longleftrightarrow \quad 4n + 6 > \frac{3}{\varepsilon}$$

$$\Longleftrightarrow \quad 4n > \frac{3}{\varepsilon} - 6$$

$$\Longleftrightarrow \quad n > \frac{3}{4\varepsilon} - \frac{6}{4} .$$

Looking at the above equivalences, it is clear that $N \in \mathbb{N}$ has to be chosen so that

$$N > \frac{3}{4\varepsilon} - \frac{6}{4} .$$

Part 2. Formal Proof. We have to show that

$$\forall \varepsilon > 0, \, \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N, \, \left| \frac{n}{2n+3} - \frac{1}{2} \right| < \varepsilon.$$

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$N > \frac{3}{4\varepsilon} - \frac{6}{4}. \tag{6.5}$$

By the rough work shown above, inequality (6.5) is equivalent to

$$\frac{3}{4N+6} < \varepsilon.$$

Let $n \geq N$. Then

$$\left| \frac{n}{2n+3} - \frac{1}{2} \right| = \left| \frac{-3}{2(2n+3)} \right|$$
$$= \frac{3}{4n+6}$$
$$\leq \frac{3}{4N+6}$$
$$< \varepsilon,$$

where in the third line we used that $n \geq N$.

We conclude by showing that constant sequences always converge.

Theorem 6.12

Let $c \in \mathbb{R}$ and define the constant sequence

$$a_n := c, \quad \forall n \in \mathbb{N}.$$

We have that

$$\lim_{n \to \infty} a_n = c.$$

Proof

We have to prove that

$$\forall \varepsilon > 0, \, \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N, \, |a_n - c| < \varepsilon. \tag{6.6}$$

Let $\varepsilon > 0$. We have

$$|a_n - c| = |c - c| = 0 < \varepsilon, \quad \forall n \in \mathbb{N}.$$

Therefore we can choose $N = 1$ and (6.6) is satisfied.

6.3 Divergent sequences

The opposite of convergent sequences are divergent sequences.

Definition 6.13:   Divergent sequence

We say that a sequence $(a_n)_{n \in \mathbb{N}}$ in $\mathbb{R}$ is **divergent** if it is not convergent.

Remark 6.14

Proving that a sequence (a_n) is divergent is more complicated than showing it is convergent: To show that (a_n) is divergent, we need to show that (a_n) cannot converge to a for any $a \in \mathbb{R}$.

In other words, we have to show that there does not exist an $a \in \mathbb{R}$ such that

$$\lim_{n \to \infty} a_n = a \,.$$

Using quantifiers, this means

$$\nexists\, a \in \mathbb{R} \ \text{s.t.} \ \forall \varepsilon > 0, \exists\, N \in \mathbb{N} \ \text{s.t.} \ \forall n \geq N, |a_n - a| < \varepsilon \,.$$

The above is equivalent to showing that

$$\forall a \in \mathbb{R}, \exists \varepsilon > 0 \ \text{s.t.} \ \forall N \in \mathbb{N}, \exists n \geq N \ \text{s.t.} \ |a_n - a| \geq \varepsilon \,.$$

Theorem 6.15

Let (a_n) be the sequence defined by

$$a_n = (-1)^n \,.$$

Then (a_n) does not converge.

Proof

To prove that (a_n) does not converge, we have to show that

$$\forall a \in \mathbb{R}, \exists \varepsilon > 0 \ \text{s.t.} \ \forall N \in \mathbb{N}, \exists n \geq N \ \text{s.t.} \ |a_n - a| \geq \varepsilon \,.$$

Let $a \in \mathbb{R}$. Choose

$$\varepsilon = \frac{1}{2} \,.$$

Let $N \in \mathbb{N}$. We distinguish two cases:

- $a \geq 0$: Choose $n = 2N + 1$. Note that $n \geq N$. Then

$$\begin{aligned}
|a_n - a| &= |a_{2N+1} - a| \\
&= \left|(-1)^{2N+1} - a\right| \\
&= |-1 - a| \\
&= 1 + a \\
&\geq 1 \\
&> \frac{1}{2} = \varepsilon \,,
\end{aligned}$$

where we used that $a \geq 0$, and therefore

$$|-1 - a| = 1 + a \geq 1 \,.$$

- $a < 0$: Choose $n = 2N$. Note that $n \geq N$. Then

$$\begin{aligned}
|a_n - a| &= |a_{2N} - a| \\
&= \left|(-1)^{2N} - a\right| \\
&= |1 - a| \\
&= 1 - a \\
&> 1 \\
&> \frac{1}{2} = \varepsilon \,,
\end{aligned}$$

where we used that $a < 0$, and therefore

$$|1 - a| = 1 - a > 1 \,.$$

6.4 Uniqueness of limit

In Definition 6.5 of convergence, we used the notation

$$\lim_{n \to \infty} a_n = a \,.$$

The above notation makes sense only if the limit is unique, that is, if we do not have that

$$\lim_{n \to \infty} a_n = b \,,$$

for some

$$a \neq b \,.$$

In the next theorem we will show that the limit is unique, if it exists.

Theorem 6.16:   Uniqueness of limit

Let $(a_n)_{n \in \mathbb{N}}$ be a sequence. Suppose that

$$\lim_{n \to \infty} a_n = a \,, \quad \lim_{n \to \infty} a_n = b \,.$$

Then $a = b$.

Proof

Assume that,

$$\lim_{n \to \infty} a_n = a \,, \quad \lim_{n \to \infty} a_n = b \,.$$

Suppose by contradiction that

$$a \neq b \,.$$

Choose

$$\varepsilon := \frac{1}{2} |a - b| \,.$$

Therefore $\varepsilon > 0$, since $|a - b| > 0$. By the convergence $a_n \to a$,

$$\exists \, N_1 \in \mathbb{N} \ \text{s.t.} \ \forall \, n \geq N_1 \,, \, |a_n - a| < \varepsilon \,.$$

By the convergence $a_n \to b$,

$$\exists \, N_2 \in \mathbb{N} \ \text{s.t.} \ \forall \, n \geq N_2 \,, \, |a_n - b| < \varepsilon \,.$$

Define

$$N := \max\{N_1, N_2\} \,.$$

Choose an $n \in \mathbb{N}$ such that $n \geq N$. In particular

$$n \geq N_1 \,, \quad n \geq N_2 \,.$$

Then

$$
\begin{aligned}
2\varepsilon = |a - b| \\
= |a - a_n + a_n - b| \\
\leq |a - a_n| + |a_n - b| \\
< \varepsilon + \varepsilon \\
= 2\varepsilon,
\end{aligned}
$$

where we used the triangle inequality in the first inequality. Hence $2\varepsilon < 2\varepsilon$, which gives a contradiction.

Example 6.17

Prove that

$$
\lim_{n \to \infty} \frac{n^2 - 1}{2n^2 - 3} = \frac{1}{2}
$$

According to Theorem 6.16, it suffices to show that the sequence

$$
\left(\frac{n^2 - 1}{2n^2 - 3} \right)_{n \in \mathbb{N}}
$$

converges to $\frac{1}{2}$, since then $\frac{1}{2}$ can be the only limit.
Part 1. Rough Work.
Let $\varepsilon > 0$. We want to find $N \in \mathbb{N}$ such that

$$
\left| \frac{n^2 - 1}{2n^2 - 3} - \frac{1}{2} \right| < \varepsilon, \quad \forall n \geq N.
$$

To this end, we compute:

$$
\begin{aligned}
\left| \frac{n^2 - 1}{2n^2 - 3} - \frac{1}{2} \right| &= \left| \frac{2(n^2 - 1) - (2n^2 - 3)}{2(2n^2 - 3)} \right| \\
&= \left| \frac{1}{4n^2 - 6} \right| \\
&= \frac{1}{4n^2 - 6} \\
&= \frac{1}{3n^2 + n^2 - 6} \\
&\leq \frac{1}{3n^2}
\end{aligned}
$$

which holds if $n \geq 3$, since in this case $n^2 - 6 \geq 0$. Therefore

$$\left| \frac{n^2 - 1}{2n^2 - 3} - \frac{1}{2} \right| < \varepsilon \quad \Longleftarrow \quad \frac{1}{3n^2} < \varepsilon$$

$$\Longleftrightarrow \quad 3n^2 > \frac{1}{\varepsilon}$$

$$\Longleftrightarrow \quad n^2 > \frac{1}{3\varepsilon}$$

$$\Longleftrightarrow \quad n > \frac{1}{\sqrt{3\varepsilon}} \, .$$

Looking at the above implications, it is clear that $N \in \mathbb{N}$ has to be chosen so that

$$N > \frac{1}{\sqrt{3\varepsilon}} \, .$$

Moreover we need to recall that N has to satisfy

$$N \geq 3$$

for the estimates to hold.

Part 2. Formal Proof. We have to show that

$$\forall \varepsilon > 0 \, , \, \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N \, , \, \left| \frac{n^2 - 1}{2n^2 - 3} - \frac{1}{2} \right| < \varepsilon \, .$$

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$N > \max \left\{ \frac{1}{\sqrt{3\varepsilon}}, 3 \right\} \, .$$

Let $n \geq N$. Then

$$\left| \frac{n^2 - 1}{2n^2 - 3} - \frac{1}{2} \right| = \frac{1}{4n^2 - 6}$$

$$= \frac{1}{3n^2 + n^2 - 6}$$

$$\leq \frac{1}{3n^2}$$

$$\leq \frac{1}{3N^2}$$

$$< \varepsilon \, ,$$

where we used that

$$n \geq N \geq 3$$

which implies

$$n^2 - 6 \geq 0 \, ,$$

in the third line. The last inequality holds, since it is equivalent to

$$N > \frac{1}{\sqrt{3\varepsilon}} \, .$$

6.5 Bounded sequences

An important property of sequences is boundedness.

Definition 6.18:   Bounded sequence

A sequence $(a_n)_{n \in \mathbb{N}}$ is called **bounded** if there exists a constant $M \in \mathbb{R}$, with $M > 0$, such that

$$|a_n| \leq M, \quad \forall\, n \in \mathbb{N}.$$

Definition 6.18 says that a sequence is bounded, if we can find some constant $M > 0$ (possibly very large), such that for all elements of the sequence it holds that

$$|a_n| \leq M,$$

or equivalently, that

$$-M \leq a_n \leq M.$$

We now show that any sequence that converges is also bounded

Theorem 6.19

If a sequence $(a_n)_{n \in \mathbb{N}}$ converges, then the sequence is bounded.

Proof

Suppose the sequence $(a_n)_{n \in \mathbb{N}}$ converges and let

$$a := \lim_{n \to \infty} a_n$$

By definition of convergence we have that

$$\forall\, \varepsilon > 0, \, \exists N \in \mathbb{N} \text{ s.t. } \forall\, n \geq N, \, |a_n - a| < \varepsilon.$$

In particular, we can choose

$$\varepsilon = 1$$

and let $N \in \mathbb{N}$ be that value such that

$$|a_n - a| < 1, \quad \forall\, n \geq N.$$

If $n \geq N$ we have, by the triangle inequality,

$$\begin{aligned}
|a_n| &= |a_n - a + a| \\
&\leq |a_n - a| + |a| \\
&< 1 + |a|.
\end{aligned}$$

Set
$$M := \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, 1 + |a|\}.$$
Note that such maximum exists, being the set finite. Then
$$|a_n| \leq M, \quad \forall\, n \in \mathbb{N},$$
showing that (a_n) is bounded.

The choice of M in the above proof says that the sequence can behave wildly for a finite number of terms. After that, it will stay close to the value of the limit, if the latter exists.

Example 6.20

In Theorem 6.8 we have shown that
$$\lim_{n\to\infty} \frac{1}{n} = 0$$
Hence, it follows from Theorem 6.19 that the sequence $(1/n)$ is bounded. Indeed, we have that
$$\left|\frac{1}{n}\right| = \frac{1}{n} \leq 1, \quad \forall\, n \in \mathbb{N},$$
since $n \geq 1$ for all $n \in \mathbb{N}$.

Warning

The converse of Theorem 6.19 does not hold: There exist sequences (a_n) which are bounded, but not convergent.

Example 6.21

Define the sequence
$$a_n = (-1)^n.$$
We have proven in Theorem 6.15 that (a_n) is not convergent. However (a_n) is bounded, with $M = 1$, since
$$|a_n| = |(-1)^n| = 1 = M, \quad \forall\, n \in \mathbb{N}.$$

Taking the contrapositive of the statement in Theorem 6.19 we get the following corollary:

Corollary 6.22

If a sequence $(a_n)_{n\in\mathbb{N}}$ is not bounded, then the sequence does not converge.

Remark 6.23

For a sequence (a_n) to be unbounded, it means that

$$\forall\, M > 0\,, \ \exists\, n \in \mathbb{N} \ \text{s.t.} \ |a_n| > M\,.$$

The above is saying that no real number $M > 0$ can be a bound for $|a_n|$, since there is always an index $n \in \mathbb{N}$ such that

$$|a_n| > M\,.$$

We can use Corollary 6.22 to show that certain sequences do not converge.

Theorem 6.24

For all $p > 0$, the sequence $(n^p)_{n\in\mathbb{N}}$ does not converge.

Proof

Let $p > 0$. We prove that the sequence $(n^p)_{n\in\mathbb{N}}$ is unbounded, that is,

$$\forall\, M > 0\,, \ \exists\, n \in \mathbb{N} \ \text{s.t.} \ |a_n| > M\,.$$

To this end, let $M > 0$. Choose $n \in \mathbb{N}$ such that

$$n > M^{1/p}\,.$$

Then

$$a_n = n^p > \left(M^{1/p}\right)^p = M\,.$$

This proves that the sequence (n^p) is unbounded. Hence (n^p) cannot converge, by Corollary 6.22.

Theorem 6.25

The sequence $(\log n)_{n\in\mathbb{N}}$ does not converge.

Proof

Let us show that $(\log n)_{n \in \mathbb{N}}$ is unbounded, that is,

$$\forall\, M > 0\,, \;\; \exists\, n \in \mathbb{N} \;\; \text{s.t.} \;\; |a_n| > M\,.$$

To this end let $M > 0$. Choose $n \in \mathbb{N}$ such that

$$n \geq e^{M+1}\,.$$

Then

$$|a_n| = |\log n| \geq \left|\log e^{M+1}\right| = M + 1 > M\,.$$

This proves that the sequence $(\log n)$ is unbounded. Hence $(\log n)$ cannot converge, by Corollary 6.22.

6.6　Algebra of limits

Proving convergence using Definition 6.5 can be a tedious task. In this section we discuss how to prove convergence, starting from known convergence results.

Theorem 6.26:　Algebra of limits

Let $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ be sequences in $\mathbb{R}$. Suppose that

$$\lim_{n \to \infty} a_n = a\,, \quad \lim_{n \to \infty} b_n = b\,,$$

for some $a, b \in \mathbb{R}$. Then,

1. Limit of sum is the sum of limits:
$$\lim_{n \to \infty} (a_n \pm b_n) = a \pm b$$

2. Limit of product is the product of limits:
$$\lim_{n \to \infty} (a_n b_n) = ab$$

3. If $b_n \neq 0$ for all $n \in \mathbb{N}$ and $b \neq 0$, then
$$\lim_{n \to \infty} \left(\frac{a_n}{b_n}\right) = \frac{a}{b}$$

Proof

Let $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ be sequences in $\mathbb{R}$ and let $c \in \mathbb{R}$. Suppose that, for some $a, b \in \mathbb{R}$

$$\lim_{n \to \infty} a_n = a, \quad \lim_{n \to \infty} b_n = b.$$

Proof of Point 1.
We need to show that

$$\lim_{n \to \infty} (a_n \pm b_n) = a \pm b.$$

We only give a proof of the formula with $+$, since the case with $-$ follows with a very similar proof. Hence, we need to show that

$$\forall \, \varepsilon > 0 \, , \, \exists \, N \in \mathbb{N} \text{ s.t. } \forall n \geq N \, , \, |(a_n + b_n) - (a + b)| < \varepsilon \, .$$

Let $\varepsilon > 0$ and set

$$\tilde{\varepsilon} := \frac{\varepsilon}{2} \, .$$

Since $a_n \to a, b_n \to b$, and $\tilde{\varepsilon} > 0$, there exist $N_1, N_2 \in \mathbb{N}$ such that

$$|a_n - a| < \tilde{\varepsilon}, \quad \forall n \geq N_1 \, ,$$
$$|b_n - b| < \tilde{\varepsilon}, \quad \forall n \geq N_2 \, .$$

Define

$$N := \max\{N_1, N_2\} \, .$$

For all $n \geq N$ we have, by the triangle inequality,

$$
\begin{aligned}
|(a_n + b_n) - (a + b)| &= |(a_n - a) + (b_n - b)| \\
&\leq |a_n - a| + |b_n - b| \\
&< \tilde{\varepsilon} + \tilde{\varepsilon} \\
&= \varepsilon \, .
\end{aligned}
$$

Proof of Point 2.
We need to show that

$$\lim_{n \to \infty} (a_n b_n) = ab \, ,$$

which is equivalent to

$$\forall \, \varepsilon > 0 \, , \, \exists \, N \in \mathbb{N} \text{ s.t. } \forall n \geq N \, , \, |a_n b_n - ab| < \varepsilon \, .$$

Let $\varepsilon > 0$. The sequence (a_n) converges, and hence is bounded, by Theorem 6.19. This means there exists some $M > 0$ such that

$$|a_n| \leq M, \quad \forall n \in \mathbb{N} \, .$$

Define

$$\tilde{\varepsilon} = \frac{\varepsilon}{M + |b|} \, .$$

Since $a_n \to a, b_n \to b$, and $\tilde{\varepsilon} > 0$, there exist $N_1, N_2 \in \mathbb{N}$ such that

$$|a_n - a| < \tilde{\varepsilon}, \quad \forall\, n \geq N_1\,.$$
$$|b_n - b| < \tilde{\varepsilon}, \quad \forall\, n \geq N_2\,.$$

Let

$$N := \max\{N_1, N_2\}\,.$$

For all $n \geq N$ we have

$$
\begin{aligned}
|a_n b_n - ab| &= |a_n b_n - a_n b + a_n b - ab| \\
&\leq |a_n b_n - a_n b| + |a_n b - ab| \\
&= |a_n|\,|b_n - b| + |b|\,|a_n - a| \\
&< M\,\tilde{\varepsilon} + |b|\,\tilde{\varepsilon} \\
&= (M + |b|)\,\tilde{\varepsilon} \\
&= \varepsilon\,.
\end{aligned}
$$

Proof of Point 3.
Suppose in addition that $b_n \neq 0$ and $b \neq 0$. We need to show that

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{a}{b}\,,$$

which is equivalent to

$$\forall\, \varepsilon > 0\,,\ \exists\, N \in \mathbb{N}\ \text{s.t.}\ \forall\, n \geq N\,,\ \left|\frac{a_n}{b_n} - \frac{a}{b}\right| < \varepsilon\,.$$

We suppose in addition that $b > 0$. The proof is very similar for the case $b < 0$, and is hence omitted. Let $\varepsilon > 0$. Set

$$\delta := \frac{b}{2}\,.$$

Since $b_n \to b$ and $\delta > 0$, there exists $N_1 \in \mathbb{N}$ such that

$$|b_n - b| < \delta \quad \forall\, n \geq N_1\,.$$

In particular we have

$$b_n > b - \delta = b - \frac{b}{2} = \frac{b}{2} \quad \forall\, n \geq N_1\,.$$

Define

$$\tilde{\varepsilon} := \frac{b^2}{2(b + |a|)}\,\varepsilon\,.$$

Since $\tilde{\varepsilon} > 0$ and $a_n \to a, b_n \to b$, there exist $N_2, N_3 \in \mathbb{N}$ such that

$$|a_n - a| < \tilde{\varepsilon}, \quad \forall\, n \geq N_2\,,$$
$$|b_n - b| < \tilde{\varepsilon}, \quad \forall\, n \geq N_3\,.$$

Define
$$N := \max\{N_1, N_2, N_3\}.$$

For all $n \geq N$ we have

$$\left| \frac{a_n}{b_n} - \frac{a}{b} \right| = \left| \frac{a_n b - a b_n}{b_n b} \right|$$

$$= \frac{1}{|b_n b|} \, |a_n b - ab + ab - ab_n|$$

$$= \frac{1}{|b_n b|} \, |(a_n - a)b + a(b - b_n)|$$

$$\leq \frac{1}{|b_n b|} \, (|a_n - a||b| + |a||b - b_n|)$$

$$< \frac{1}{\frac{b}{2} b} \, (\tilde{\varepsilon} \, b + \tilde{\varepsilon} |a|)$$

$$= \frac{2(b + |a|)}{b^2} \, \tilde{\varepsilon}$$

$$= \varepsilon \, .$$

In the future we will refer to Theorem 6.26 as the *Algebra of Limits*. We now show how to use Theorem 6.26 for computing certain limits.

Example 6.27

Prove that
$$\lim_{n \to \infty} \frac{3n}{7n + 4} = \frac{3}{7} \, .$$

Proof. We can rewrite
$$\frac{3n}{7n + 4} = \frac{3}{7 + \dfrac{4}{n}}$$

By Theorem 6.12 we know that
$$3 \to 3 \, , \quad 4 \to 4 \, , \quad 7 \to 7 \, .$$

From Theorem 6.8 we know that
$$\frac{1}{n} \to 0 \, .$$

Hence, it follows from Theorem 6.26 Point 2 that
$$\frac{4}{n} = 4 \cdot \frac{1}{n} \to 4 \cdot 0 = 0 \, .$$

By Theorem 6.26 Point 1 we have
$$7 + \frac{4}{n} \to 7 + 0 = 7 \, .$$

Finally we can use Theorem 6.26 Point 3 to infer

$$\frac{3n}{7n+4} = \frac{3}{7 + \dfrac{4}{n}} \to \frac{3}{7}\,.$$

Important

The technique shown in Example 6.27 is useful to compute limits of fractions of polynomials. To identify the possible limit, if it exists, it is often best to divide by the largest power of n in the denominator.

Example 6.28

Prove that

$$\lim_{n\to\infty} \frac{n^2-1}{2n^2-3} = \frac{1}{2}\,.$$

Proof. Factor n^2 to obtain

$$\frac{n^2-1}{2n^2-3} = \frac{1 - \dfrac{1}{n^2}}{2 - \dfrac{3}{n^2}}\,.$$

By Theorem 6.9 we have

$$\frac{1}{n^2} \to 0\,.$$

We can then use the Algebra of Limits Theorem 6.26 Point 2 to infer

$$\frac{3}{n^2} \to 3 \cdot 0 = 0$$

and Theorem 6.26 Point 1 to get

$$1 - \frac{1}{n^2} \to 1 - 0 = 1\,, \quad 2 - \frac{3}{n^2} \to 2 - 0 = 2\,.$$

Finally we use Theorem 6.26 Point 3 and conclude

$$\frac{1 - \dfrac{1}{n^2}}{2 - \dfrac{3}{n^2}} \to \frac{1}{2}\,.$$

Therefore

$$\lim_{n\to\infty} \frac{n^2-1}{2n^2-3} = \lim_{n\to\infty} \frac{1 - \dfrac{1}{n^2}}{2 - \dfrac{3}{n^2}} = \frac{1}{2}\,.$$

We can also use the Algebra of Limits to prove that certain limits do not exist.

Example 6.29

Prove that the sequence

$$a_n = \frac{4n^3 + 8n + 1}{7n^2 + 2n + 1}$$

does not converge.

Proof. To show that the sequence (a_n) does not converge, we divide by the largest power in the denominator, which in this case is n^2

$$
a_n = \frac{4n^3 + 8n + 1}{7n^2 + 2n + 1}
$$
$$
= \frac{4n + \dfrac{8}{n} + \dfrac{1}{n^2}}{7 + \dfrac{2}{n} + \dfrac{1}{n^2}}
$$
$$
= \frac{b_n}{c_n}
$$

where we set

$$b_n := 4n + \frac{8}{n} + \frac{1}{n^2}, \quad c_n := 7 + \frac{2}{n} + \frac{1}{n^2}.$$

Using the Algebra of Limits Theorem 6.26 we see that

$$c_n = 7 + \frac{2}{n} + \frac{1}{n^2} \to 7.$$

Suppose by contradiction that

$$a_n \to a$$

for some $a \in \mathbb{R}$. Then, by the Algebra of Limits Theorem 6.26 we would infer

$$b_n = c_n \cdot a_n \to 7a,$$

concluding that b_n is convergent to $7a$. We have that

$$b_n = 4n + d_n, \quad d_n := \frac{8}{n} + \frac{1}{n^2}.$$

Again by the Algebra of Limits Theorem 6.26 we get that

$$d_n = \frac{8}{n} + \frac{1}{n^2} \to 0,$$

and hence

$$4n = b_n - d_n \to 7a - 0 = 7a.$$

This is a contradiction, since the sequence $(4n)$ is unbounded, and hence cannot be convergent. Hence (a_n) is not convergent.

Warning

Consider the sequence

$$a_n = \frac{4n^3 + 8n + 1}{7n^2 + 2n + 1}$$

from the previous example. We have proven that (a_n) is not convergent, by making use of the Algebra of Limits.

Let us review a **faulty** argument to conclude that (a_n) is not convergent. Write

$$a_n = \frac{b_n}{c_n}, \quad b_n := 4n^3 + 8n + 1, \quad c_n := 7n^2 + 2n + 1.$$

The numerator

$$b_n = 4n^3 + 8n + 1$$

and denominator

$$c_n = 7n^2 + 2n + 1$$

are both unbounded, and hence (b_n) and (c_n) do not converge. One might be tempted to conclude that (a_n) does not converge. However this is **false** in general: as seen in Example 6.28, we have

$$\lim_{n \to \infty} \frac{n^2 - 1}{2n^2 - 3} = \frac{1}{2},$$

while numerator and denominator are unbounded.

Sometimes it is useful to rearrange the terms of a sequence, before applying the Algebra of Limits.

Example 6.30

Define

$$a_n := \frac{2n^3 + 7n + 1}{5n + 9} \cdot \frac{8n + 9}{6n^3 + 8n^2 + 3}.$$

Prove that

$$\lim_{n \to \infty} a_n = \frac{8}{15}.$$

Proof. The first fraction in (a_n) does not converge, as it is unbounded. Therefore we cannot use Point 2 in Theorem 6.26 directly. However, we note that

$$a_n = \frac{2n^3 + 7n + 1}{5n + 9} \cdot \frac{8n + 9}{6n^3 + 8n^2 + 3}$$

$$= \frac{8n + 9}{5n + 9} \cdot \frac{2n^3 + 7n + 1}{6n^3 + 8n^2 + 3}.$$

Factoring out n and n^3, respectively, and using the Algebra of Limits, we see that

$$\frac{8n+9}{5n+9} = \frac{8+9/n}{5+9/n} \to \frac{8+0}{5+0} = \frac{8}{5}$$

and

$$\frac{2+7/n^2+1/n^3}{6+8/n+3/n^3} \to \frac{2+0+0}{6+0+0} = \frac{1}{3}$$

Therefore Theorem 6.26 Point 2 ensures that

$$a_n \to \frac{8}{5} \cdot \frac{1}{3} = \frac{8}{15} \, .$$

6.7 Fractional powers

The Algebra of Limits Theorem 6.26 can also be used when fractional powers of n are involved.

Example 6.31

Prove that

$$a_n = \frac{n^{7/3} + 2\sqrt{n} + 7}{4n^{3/2} + 5n}$$

does not converge.

Proof. The largest power of n in the denominator is $n^{3/2}$. Hence we factor out $n^{3/2}$

$$\begin{aligned} a_n &= \frac{n^{7/3} + 2\sqrt{n} + 7}{4n^{3/2} + 5n} \\ &= \frac{n^{7/3-3/2} + 2n^{1/2-3/2} + 7n^{-3/2}}{4 + 5n^{-3/2}} \\ &= \frac{n^{5/6} + 2n^{-1} + 7n^{-3/2}}{4 + 5n^{-3/2}} \\ &= \frac{b_n}{c_n} \end{aligned}$$

where we set

$$b_n := n^{5/6} + 2n^{-1} + 7n^{-3/2}, \quad c_n := 4 + 5n^{-3/2} \, .$$

We see that b_n is unbounded while $c_n \to 4$. By the Algebra of Limits (and usual contradiction argument) we conclude that (a_n) is divergent.

We now present a general result about the square root of a sequence.

Theorem 6.32

Let $(a_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}$ such that

$$\lim_{n \to \infty} a_n = a \,,$$

for some $a \in \mathbb{R}$. If $a_n \geq 0$ for all $n \in \mathbb{N}$ and $a \geq 0$, then

$$\lim_{n \to \infty} \sqrt{a_n} = \sqrt{a} \,.$$

Proof

Let $\varepsilon > 0$. We the two cases $a > 0$ and $a = 0$:

- $a > 0$: Define

$$\delta := \frac{a}{2} \,.$$

Since $\delta > 0$ and $a_n \to a$, there exists $N_1 \in \mathbb{N}$ such that

$$|a_n - a| < \delta \,, \quad \forall\, n \geq N_1 \,.$$

In particular

$$a_n > a - \delta = a - \frac{a}{2} = \frac{a}{2} \,, \quad \forall\, n \geq N_1 \,,$$

from which we infer

$$\sqrt{a_n} > \sqrt{a/2} \,, \quad \forall\, n \geq N_1 \,,$$

Now set

$$\tilde{\varepsilon} := \left(\sqrt{a/2} + \sqrt{a} \right) \varepsilon \,.$$

Since $\tilde{\varepsilon} > 0$ and $a_n \to a$, there exists $N_2 \in \mathbb{N}$ such that

$$|a_n - a| < \tilde{\varepsilon} \,, \quad \forall\, n \geq N_2 \,.$$

Let

$$N := \max\{N_1, N_2\} \,.$$

For $n \geq N$ we have

$$
\begin{aligned}
\left| \sqrt{a_n} - \sqrt{a} \right| &= \left| \frac{\left(\sqrt{a_n} - \sqrt{a} \right) \left(\sqrt{a_n} + \sqrt{a} \right)}{\sqrt{a_n} + \sqrt{a}} \right| \\
&= \frac{|a_n - a|}{\sqrt{a_n} + \sqrt{a}} \\
&< \frac{\tilde{\varepsilon}}{\sqrt{a/2} + \sqrt{a}} \\
&= \varepsilon \,.
\end{aligned}
$$

- $a = 0$: In this case

$$a_n \to a = 0 .$$

Since $\varepsilon^2 > 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n - 0| = |a_n| < \varepsilon^2 , \quad \forall\, n \geq N .$$

Therefore

$$\left|\sqrt{a_n} - \sqrt{0}\right| = \left|\sqrt{a_n}\right| < \sqrt{\varepsilon^2} = \varepsilon , \quad \forall\, n \geq N .$$

Let us show an application of Theorem 6.32.

Example 6.33

Define the sequence

$$a_n = \sqrt{9n^2 + 3n + 1} - 3n .$$

Prove that

$$\lim_{n \to \infty} a_n = \frac{1}{2} .$$

Proof. We first rewrite

$$
\begin{aligned}
a_n &= \sqrt{9n^2 + 3n + 1} - 3n \\
&= \frac{\left(\sqrt{9n^2 + 3n + 1} - 3n\right)\left(\sqrt{9n^2 + 3n + 1} + 3n\right)}{\sqrt{9n^2 + 3n + 1} + 3n} \\
&= \frac{9n^2 + 3n + 1 - (3n)^2}{\sqrt{9n^2 + 3n + 1} + 3n} \\
&= \frac{3n + 1}{\sqrt{9n^2 + 3n + 1} + 3n} .
\end{aligned}
$$

The biggest power of n in the denominator is n. Therefore we factor out n:

$$
\begin{aligned}
a_n &= \sqrt{9n^2 + 3n + 1} - 3n \\
&= \frac{3n + 1}{\sqrt{9n^2 + 3n + 1} + 3n} \\
&= \frac{3 + \dfrac{1}{n}}{\sqrt{9 + \dfrac{3}{n} + \dfrac{1}{n^2}} + 3} .
\end{aligned}
$$

By the Algebra of Limits we have

$$9 + \frac{3}{n} + \frac{1}{n^2} \to 9 + 0 + 0 = 9 .$$

Therefore we can use Theorem 6.32 to infer

$$\sqrt{9 + \frac{3}{n} + \frac{1}{n^2}} \to \sqrt{9}\,.$$

By the Algebra of Limits we conclude:

$$a_n = \frac{3 + \dfrac{1}{n}}{\sqrt{9 + \dfrac{3}{n} + \dfrac{1}{n^2}} + 3} \to \frac{3 + 0}{\sqrt{9} + 3} = \frac{1}{2}\,.$$

Example 6.34

Prove that the sequence

$$a_n = \sqrt{9n^2 + 3n + 1} - 2n$$

does not converge.

Proof. We rewrite a_n as

$$
\begin{aligned}
a_n &= \sqrt{9n^2 + 3n + 1} - 2n \\
&= \frac{(\sqrt{9n^2 + 3n + 1} - 2n)(\sqrt{9n^2 + 3n + 1} + 2n)}{\sqrt{9n^2 + 3n + 1} + 2n} \\
&= \frac{9n^2 + 3n + 1 - (2n)^2}{\sqrt{9n^2 + 3n + 1} + 2n} \\
&= \frac{5n^2 + 3n + 1}{\sqrt{9n^2 + 3n + 1} + 2n} \\
&= \frac{5n + 3 + \dfrac{1}{n}}{\sqrt{9 + \dfrac{3}{n} + \dfrac{1}{n^2}} + 2} \\
&= \frac{b_n}{c_n}\,,
\end{aligned}
$$

where we factored n, being it the largest power of n in the denominator, and defined

$$b_n := 5n + 3 + \frac{1}{n}\,, \quad c_n := \sqrt{9 + \frac{3}{n} + \frac{1}{n^2}} + 2\,.$$

Note that

$$9 + \frac{3}{n} + \frac{1}{n^2} \to 9$$

by the Algebra of Limits. Therefore

$$\sqrt{9 + \frac{3}{n} + \frac{1}{n^2}} \to \sqrt{9} = 3$$

by Theorem 6.32. Hence

$$c_n = \sqrt{9 + \frac{3}{n} + \frac{1}{n^2}} + 2 \to 3 + 2 = 5 \,.$$

The numerator

$$b_n = 5n + 3 + \frac{1}{n}$$

is instead unbounded. Therefore (a_n) is not convergent, by the Algebra of Limits and the usual contradiction argument.

6.8 Limit Tests

In this section we discuss a number of *Tests* to determine whether a sequence converges or not. These are known as **Limit Tests**.

6.8.1 Squeeze Theorem

When a sequence (a_n) oscillates, it is difficult to compute the limit. Examples of terms which produce oscillations are

$$(-1)^n, \quad \sin(n), \quad \cos(n) \,.$$

In such instance it might be useful to compare (a_n) with other sequences whose limit is known. If we can prove that (a_n) is *squeezed* between two other sequences with the same limiting value, then we can show that also (a_n) converges to this value.

Theorem 6.35: Squeeze theorem

Let $(a_n), (b_n)$ and (c_n) be sequences in $\mathbb{R}$. Suppose that

$$b_n \le a_n \le c_n, \quad \forall\, n \in \mathbb{N} \,,$$

and that

$$\lim_{n \to \infty} b_n = \lim_{n \to \infty} c_n = L \,.$$

Then

$$\lim_{n\to\infty} a_n = L.$$

Proof

Let $\varepsilon > 0$. Since $b_n \to L$ and $c_n \to L$, there exist $N_1, N_2 \in \mathbb{N}$ such that

$$-\varepsilon < b_n - L < \varepsilon, \quad \forall\, n \geq N_1,$$
$$-\varepsilon < c_n - L < \varepsilon, \quad \forall\, n \geq N_2.$$

Set

$$N := \max\{N_1, N_2\}.$$

Let $n \geq N$. Using the assumption that $b_n \leq a_n \leq c_n$, we get

$$b_n - L \leq a_n - L \leq c_n - L.$$

In particular

$$-\varepsilon < b_n - L \leq a_n - L \leq b_n - L < \varepsilon.$$

The above implies

$$-\varepsilon < a_n - L < \varepsilon \quad \implies \quad |a_n - L| < \varepsilon.$$

Example 6.36

Prove that

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0.$$

Proof. For all $n \in \mathbb{N}$ we can estimate

$$-1 \leq (-1)^n \leq 1.$$

Therefore

$$\frac{-1}{n} \leq \frac{(-1)^n}{n} \leq \frac{1}{n}, \quad \forall\, n \in \mathbb{N}.$$

Moreover

$$\lim_{n\to\infty} \frac{-1}{n} = -1 \cdot 0 = 0, \quad \lim_{n\to\infty} \frac{1}{n} = 0.$$

By the Squeeze Theorem 6.35 we conclude

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0.$$

Example 6.37

Prove that

$$\lim_{n \to \infty} \frac{\cos(3n) + 9n^2}{11n^2 + 15 \sin(17n)} = \frac{9}{11} \,.$$

Proof. We know that

$$-1 \leq \cos(x) \leq 1 \,, \quad -1 \leq \sin(x) \leq 1 \,, \quad \forall\, x \in \mathbb{R} \,.$$

Therefore, for all $n \in \mathbb{N}$

$$-1 \leq \cos(3n) \leq 1 \,, \quad -1 \leq \sin(17n) \leq 1 \,.$$

We can use the above to estimate the numerator in the given sequence:

$$-1 + 9n^2 \leq \cos(3n) + 9n^2 \leq 1 + 9n^2 \,. \tag{6.7}$$

Concerning the denominator, we have

$$11n^2 - 15 \leq 11n^2 + 15 \sin(17n) \leq 11n^2 + 15$$

and therefore

$$\frac{1}{11n^2 + 15} \leq \frac{1}{11n^2 + 15 \sin(17n)} \leq \frac{1}{11n^2 - 15} \,. \tag{6.8}$$

Putting together (6.7)-(6.8) we obtain

$$\frac{-1 + 9n^2}{11n^2 + 15} \leq \frac{\cos(3n) + 9n^2}{11n^2 + 15 \sin(17n)} \leq \frac{1 + 9n^2}{11n^2 - 15} \,.$$

By the Algebra of Limits we infer

$$\frac{-1 + 9n^2}{11n^2 + 15} = \frac{-\dfrac{1}{n^2} + 9}{11 + \dfrac{15}{n^2}} \to \frac{0 + 9}{11 + 0} = \frac{9}{11}$$

and

$$\frac{1 + 9n^2}{11n^2 - 15} = \frac{\dfrac{1}{n^2} + 9}{11 - \dfrac{15}{n^2}} \to \frac{0 + 9}{11 + 0} = \frac{9}{11} \,.$$

Applying the Squeeze Theorem 6.35 we conclude

$$\lim_{n \to \infty} \frac{\cos(3n) + 9n^2}{11n^2 + 15 \sin(17n)} = \frac{9}{11} \,.$$

Warning

Suppose that the sequences $(a_n), (b_n), (c_n)$ satisfy

$$b_n \leq a_n \leq c_n, \quad \forall n \in \mathbb{N},$$

and

$$b_n \to L_1, \quad c_n \to L_2, \quad L_1 \neq L_2.$$

In general, we cannot conclude that a_n converges.

Example 6.38

Consider the sequence

$$a_n = \left(1 + \frac{1}{n}\right)(-1)^n.$$

For all $n \in \mathbb{N}$ we can bound

$$-1 - \frac{1}{n} \leq \left(1 + \frac{1}{n}\right)(-1)^n \leq 1 + \frac{1}{n}.$$

However

$$-1 - \frac{1}{n} \longrightarrow -1 - 0 = -1$$

and

$$1 + \frac{1}{n} \longrightarrow 1 + 0 = 1.$$

Since

$$-1 \neq 1,$$

we cannot apply the Squeeze Theorem 6.35 to conclude convergence of (a_n). Indeed, (a_n) is a divergent sequence.

Proof. Suppose by contradiction that $a_n \to a$. We have

$$a_n = (-1)^n + \frac{(-1)^n}{n} = b_n + c_n$$

where

$$b_n := (-1)^n, \quad c_n := \frac{(-1)^n}{n}.$$

We have seen in Example 6.37 that $c_n \to 0$. Therefore, by the Algebra of Limits, we have

$$b_n = a_n - c_n \longrightarrow a - 0 = a.$$

However, Theorem 6.15 says that the sequence $b_n = (-1)^n$ diverges. Contradiction. Hence (a_n) diverges.

6.8.2 Geometric sequences

Definition 6.39

A sequence (a_n) is called a **geometric sequence** if

$$a_n = x^n \,,$$

for some $x \in \mathbb{R}$.

The value of $|x|$ determines whether or not a geometric sequence converges, as shown in the following theorem.

Theorem 6.40: Geometric Sequence Test

Let $x \in \mathbb{R}$ and let (a_n) be the sequence defined by

$$a_n := x^n \,.$$

We have:

1. If $|x| < 1$, then
$$\lim_{n \to \infty} a_n = 0 \,.$$

2. If $|x| > 1$, then sequence (a_n) is unbounded, and hence divergent.

Warning

The Geometric Sequence Test in Theorem 6.40 does not address the case

$$|x| = 1 \,.$$

This is because, in this case, the sequence
$$a_n = x^n$$
might converge or diverge, depending on the value of x. Indeed,

$$|x| = 1 \quad \implies \quad x = \pm 1 \,.$$

We therefore have two cases:

- $x = 1$: Then
$$a_n = 1^n = 1$$
so that $a_n \to 1$ and (a_n) is convergent.

- $x = -1$: Then

$$a_n = x^n = (-1)^n$$

  which is divergent by Theorem 6.15.

To prove Theorem 6.40 we need the following inequality, known as Bernoulli's inequality.

Lemma 6.41: Bernoulli's inequality

Let $x \in \mathbb{R}$ with $x > -1$. Then

$$(1 + x)^n \geq 1 + nx, \quad \forall n \in \mathbb{N}. \tag{6.9}$$

Proof

Let $x \in \mathbb{R}, x > -1$. We prove the statement by induction:

- Base case: (6.9) holds with equality when $n = 1$.

- Induction hypothesis: Let $k \in \mathbb{N}$ and suppose that (6.9) holds for $n = k$, i.e.,

$$(1 + x)^k \geq 1 + kx.$$

  Then

$$\begin{aligned}
(1 + x)^{k+1} &= (1 + x)^k (1 + x) \\
&\geq (1 + kx)(1 + x) \\
&= 1 + kx + x + kx^2 \\
&\geq 1 + (k + 1)x,
\end{aligned}$$

  where we used that $kx^2 \geq 0$. Then (6.9) holds for $n = k + 1$.

By induction we conclude (6.9).

We are ready to prove Theorem 6.40.

Proof: Proof of Theorem 6.40

Part 1. The case $|x| < 1$.

If $x = 0$, then

$$a_n = x^n = 0$$

so that $a_n \to 0$. Hence assume $x \neq 0$. We need to prove that

$$\forall \, \varepsilon > 0 \,, \, \exists \, N \in \mathbb{N} \ \text{ s.t. } \ \forall \, n \geq N \,, \ |x^n - 0| < \varepsilon \,.$$

Let $\varepsilon > 0$. We have

$$|x| < 1 \quad \implies \quad \frac{1}{|x|} > 1 \,.$$

Therefore

$$|x| = \frac{1}{1 + u} \,, \quad u := \frac{1}{|x|} - 1 > 0 \,.$$

Let $N \in \mathbb{N}$ be such that

$$N > \frac{1}{\varepsilon u} \,,$$

so that

$$\frac{1}{Nu} < \varepsilon \,.$$

Let $n \geq N$. Then

$$|x^n - 0| = |x|^n$$
$$= \left(\frac{1}{1 + u} \right)^n$$
$$= \frac{1}{(1 + u)^n}$$
$$\leq \frac{1}{1 + nu}$$
$$\leq \frac{1}{nu}$$
$$\leq \frac{1}{Nu}$$
$$< \varepsilon \,,$$

where we used Bernoulli's inequality (6.9) in the first inequality.

Part 2. The case $|x| > 1$.

To prove that (a_n) does not converge, we prove that it is unbounded. This means showing that

$$\forall \, M > 0 \,, \, \exists \, n \in \mathbb{N} \ \text{ s.t. } \ |a_n| > M \,.$$

Let $M > 0$. We have two cases:

- $0 < M \leq 1$: Choose $n = 1$. Then

$$|a_1| = |x| > 1 \geq M \,.$$

- $M > 1$: Choose $n \in \mathbb{N}$ such that

$$n > \frac{\log M}{\log |x|}.$$

Note that $\log |x| > 0$ since $|x| > 1$. Therefore

$$n > \frac{\log M}{\log |x|} \iff n \log |x| > \log M$$
$$\iff \log |x|^n > \log M$$
$$\iff |x|^n > M.$$

Then

$$|a_n| = |x^n| = |x|^n > M.$$

Hence (a_n) is unbounded. By Corollary 6.22 we conclude that (a_n) is divergent.

Example 6.42

We can apply Theorem 6.40 to prove convergence or divergence for the following sequences.

1. We have

$$\left(\frac{1}{2}\right)^n \longrightarrow 0$$

 since

$$\left|\frac{1}{2}\right| = \frac{1}{2} < 1.$$

2. We have

$$\left(\frac{-1}{2}\right)^n \longrightarrow 0$$

 since

$$\left|\frac{-1}{2}\right| = \frac{1}{2} < 1.$$

3. The sequence

$$a_n = \left(\frac{-3}{2}\right)^n$$

 does not converge, since

$$\left|\frac{-3}{2}\right| = \frac{3}{2} > 1.$$

4. As $n \to \infty$,

$$\frac{3^n}{(-5)^n} = \left(-\frac{3}{5}\right)^n \longrightarrow 0$$

 since

$$\left|-\frac{3}{5}\right| = \frac{3}{5} < 1.$$

5. The sequence
$$a_n = \frac{(-7)^n}{2^{2n}}$$
does not converge, since
$$\frac{(-7)^n}{2^{2n}} = \frac{(-7)^n}{\left(2^2\right)^n} = \left(-\frac{7}{4}\right)^n$$
and
$$\left|-\frac{7}{4}\right| = \frac{7}{4} > 1.$$

6.8.3 Ratio Test

Theorem 6.43: Ratio Test

Let (a_n) be a sequence in $\mathbb{R}$ such that
$$a_n \neq 0, \quad \forall\, n \in \mathbb{N}.$$

1. Suppose that the following limit exists:
$$L := \lim_{n \to \infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

   Then,

   - If $L < 1$ we have
$$\lim_{n \to \infty} a_n = 0.$$

   - If $L > 1$, the sequence (a_n) is unbounded, and hence does not converge.

2. Suppose that there exists $N \in \mathbb{N}$ and $L > 1$ such that
$$\left|\frac{a_{n+1}}{a_n}\right| \geq L, \quad \forall\, n \geq N.$$

   Then the sequence (a_n) is unbounded, and hence does not converge.

Proof

Define the sequence $b_n = |a_n|$. Then,

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{|a_{n+1}|}{|a_n|} = \frac{b_{n+1}}{b_n}$$

Part 1. Suppose that there exists the limit

$$L := \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| .$$

Therefore

$$\lim_{n \to \infty} \frac{b_{n+1}}{b_n} = L . \tag{6.10}$$

- $L < 1$: Choose $r > 0$ such that

$$L < r < 1 .$$

  Set

$$\varepsilon := r - L$$

  By the convergence at (6.10) there exists $N \in \mathbb{N}$ such that

$$\left| \frac{b_{n+1}}{b_n} - L \right| < \varepsilon = r - L , \quad \forall\, n \geq N .$$

  In particular

$$\frac{b_{n+1}}{b_n} - L < r - L , \quad \forall\, n \geq N ,$$

  which implies

$$b_{n+1} < r\, b_n , \quad \forall\, n \geq N . \tag{6.11}$$

  Let $n \geq N$, we can use (6.11) recursively and obtain

$$0 \leq b_n < r b_{n-1} < ... < r^{n-N}\, b_N = r^n \frac{b_N}{r^N} .$$

  In particular, we have proven that

$$0 \leq b_n < r^n \frac{b_N}{r^N} , \quad \forall\, n \in \mathbb{N} . \tag{6.12}$$

  Since $|r| < 1$, by the Geometric Sequence Test Theorem 6.40 we infer

$$r^n \to 0 .$$

  The Algebra of Limits the yields

$$r^n \frac{b_N}{r^N} \to 0 .$$

  By the Squeeze Theorem 6.35 applied to (6.12), it follows that

$$b_n = |a_n| \to 0 .$$

Since
$$- |a_n| \le a_n \le |a_n| \,,$$
and
$$-|a_n| \to 0 \,, \quad |a_n| \to 0 \,,$$
we can again apply the Squeeze Theorem 6.35 to infer
$$a_n \to 0 \,.$$

- $L > 1$: Choose $r > 0$ such that
$$1 < r < L \,.$$

  Define
$$\varepsilon := L - r > 0 \,.$$

  By the convergence (6.10), there exists $N \in \mathbb{N}$ such that
$$\left| \frac{b_{n+1}}{b_n} - L \right| < \varepsilon = L - r \,, \quad \forall\, n \ge N \,.$$

  In particular,
$$-(L - r) < \frac{b_{n+1}}{b_n} - L \,, \quad \forall\, n \ge N \,,$$
which implies
$$b_{n+1} > r\, b_n \,, \quad \forall\, n \ge N \,. \tag{6.13}$$

  Let $n \ge N$. Applying (6.13) recursively we get
$$b_n > r^{n-N} b_N = r^n \frac{b_N}{r^N} \,, \quad \forall\, n \ge N \,. \tag{6.14}$$

  Since $|r| > 1$, by the Geometric Sequence Test we have that the sequence
$$(r^n)$$
  is unbounded. Therefore also the right hand side of (6.14) is unbounded, proving that (b_n) is unbounded. Since
$$b_n = |a_n| \,,$$
  we conclude that (a_n) is unbounded. By Corollary 6.22 we conclude that (a_n) does not converge.

Part 2. Suppose that there exists $N \in \mathbb{N}$ and $L > 1$ such that
$$\left| \frac{a_{n+1}}{a_n} \right| \ge L \,, \quad \forall\, n \ge N \,.$$

Since $b_n = |a_n|$, we infer
$$b_{n+1} \ge L\, b_n \,, \quad \forall\, n \ge N \,.$$

Arguing as above, we obtain

$$b_n \geq L^n \frac{b_N}{L^N}, \quad \forall\, n \geq N.$$

Since $L > 1$, we have that the sequence

$$L^n \frac{b_N}{L^N}$$

is unbounded, by the Geometric Sequence Test. Hence also (b_n) is unbounded, from which we conclude that (a_n) is unbounded. By Corollary 6.22 we conclude that (a_n) does not converge.

Let us apply the Ratio Test to some concrete examples.

Example 6.44

Let

$$a_n = \frac{3^n}{n!},$$

where we recall that $n!$ (pronounced n factorial) is defined by

$$n! := n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1.$$

Prove that

$$\lim_{n \to \infty} a_n = 0.$$

Proof. We have

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{\left(\dfrac{3^{n+1}}{(n+1)!} \right)}{\left(\dfrac{3^n}{n!} \right)}$$

$$= \frac{3^{n+1}}{3^n} \frac{n!}{(n+1)!}$$

$$= \frac{3 \cdot 3^n}{3^n} \frac{n!}{(n+1)n!}$$

$$= \frac{3}{n+1} \longrightarrow L = 0.$$

Hence, $L = 0 < 1$ so $a_n \to 0$ by the Ratio Test in Theorem 6.43.

Example 6.45

Consider the sequence

$$a_n = \frac{n! \cdot 3^n}{\sqrt{(2n)!}} \, .$$

Prove that (a_n) is divergent.

Proof. We have

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)! \cdot 3^{n+1}}{\sqrt{(2(n+1))!}} \frac{\sqrt{(2n)!}}{n! \cdot 3^n}$$

$$= \frac{(n+1)!}{n!} \cdot \frac{3^{n+1}}{3^n} \cdot \frac{\sqrt{(2n)!}}{\sqrt{(2(n+1))!}}$$

For the first two fractions we have

$$\frac{(n+1)!}{n!} \cdot \frac{3^{n+1}}{3^n} = 3(n+1) \, ,$$

while for the third fraction

$$\frac{\sqrt{(2n)!}}{\sqrt{(2(n+1))!}} = \sqrt{\frac{(2n)!}{(2n+2)!}}$$

$$= \sqrt{\frac{(2n)!}{(2n+2) \cdot (2n+1) \cdot (2n)!}}$$

$$= \frac{1}{\sqrt{(2n+1)(2n+2)}} \, .$$

Therefore, using the Algebra of Limits,

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{3(n+1)}{\sqrt{(2n+1)(2n+2)}}$$

$$= \frac{3n \left(1 + \dfrac{1}{n} \right)}{\sqrt{n^2 \left(2 + \dfrac{1}{n} \right) \left(2 + \dfrac{2}{n} \right)}}$$

$$= \frac{3 \left(1 + \dfrac{1}{n} \right)}{\sqrt{\left(2 + \dfrac{1}{n} \right) \left(2 + \dfrac{2}{n} \right)}} \longrightarrow \frac{3}{\sqrt{4}} = \frac{3}{2} > 1 \, .$$

By the Ratio Test we conclude that (a_n) is divergent.

Example 6.46

Let

$$a_n = \frac{n!}{100^n} \, .$$

Prove that (a_n) is divergent.

Proof.

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{100^n}{100^{n+1}} \frac{(n+1)!}{n!} = \frac{n+1}{100} \, .$$

Choose $N = 101$. Then for all $n \geq N$,

$$\left| \frac{a_{n+1}}{a_n} \right| \geq \frac{101}{100} > 1 \, .$$

Hence a_n is divergent by the Ratio Test.

Warning

The Ratio Test in Theorem 6.43 does not address the case

$$L = 1 \, .$$

This is because, in this case, the sequence (a_n) might converge or diverge.
For example:

- Define the sequence

$$a_n = \frac{1}{n} \, .$$

  We have

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{n}{n+1} \rightarrow L = 1 \, .$$

  Hence we cannot apply the Ratio Test. However we know that

$$\lim_{n \to \infty} \frac{1}{n} = 0 \, .$$

- Consider the sequence

$$a_n = n \, .$$

  We have

$$\frac{|a_{n+1}|}{|a_n|} = \frac{n+1}{n} \rightarrow L = 1 \, .$$

  Hence we cannot apply the Ratio Test. However we know that (a_n) is unbounded, and thus divergent.

If the sequence (a_n) is geometric, the Ratio Test of Theorem 6.43 will give the same answer as the Geometric

Sequence Test of Theorem 6.40. This is the content of the following remark.

Remark 6.47

Let $x \in \mathbb{R}$ and define the geometric sequence

$$a_n = x^n.$$

Then

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{|x^{n+1}|}{|x^n|} = \frac{|x|^{n+1}}{|x|^n} = |x| \to |x|.$$

Hence:

- If $|x| < 1$, the sequence (a_n) converges by the Ratio Test
- If $|x| > 1$, the sequence (a_n) diverges by the Ratio Test.
- If $|x| = 1$, the sequence (a_n) might be convergent or divergent.

These results are in agreement with the Geometric Sequence Test.

6.9 Monotone sequences

We have seen in Theorem 6.19 that convergent sequences are bounded. We noted that the converse statement is not true. For example the sequence

$$a_n = (-1)^n$$

is bounded but not convergent, as shown in Theorem 6.15. On the other hand, if a bounded sequence is **monotone**, then it is convergent.

Definition 6.48: Monotone sequence

Let (a_n) be a real sequence. We say that:

1. (a_n) is **increasing** if

$$a_n \leq a_{n+1}, \quad \forall\, n \geq N.$$

2. (a_n) is **decreasing** if

$$a_n \geq a_{n+1}, \quad \forall\, n \geq N.$$

3. (a_n) is **monotone** if it is either increasing or decreasing.

Example 6.49

- The sequence below is increasing

$$b_n = \frac{n-1}{n}.$$

We have

$$b_{n+1} = \frac{n}{n+1} > \frac{n-1}{n} = b_n,$$

where the inequality holds because

$$\frac{n}{n+1} > \frac{n-1}{n} \quad \Longleftrightarrow \quad n^2 > (n-1)(n+1)$$
$$\Longleftrightarrow \quad n^2 > n^2 - 1$$
$$\Longleftrightarrow \quad 0 > -1$$

- The sequence below is decreasing

$$a_n = \frac{1}{n}.$$

We have

$$a_n = \frac{1}{n} > \frac{1}{n+1} = a_{n+1}.$$

The main result about monotone sequences is the Monotone Convergence Theorem.

Theorem 6.50: Monotone Convergence Theorem

Let (a_n) be a sequence in $\mathbb{R}$. Suppose that (a_n) is bounded and monotone. Then (a_n) converges.

Proof

Assume (a_n) is bounded and monotone. Since (a_n) is bounded, the set

$$A := \{a_n \ : \ n \in \mathbb{N}\} \subseteq \mathbb{R}$$

is bounded below and above. By the Axiom of Completeness of $\mathbb{R}$ there exist $i, s \in \mathbb{R}$ such that

$$i = \inf A, \quad s = \sup A.$$

We have two cases:

1. (a_n) is increasing: We are going to prove that

$$\lim_{n \to \infty} a_n = s.$$

Equivalently, we need to prove that

$$\forall \varepsilon > 0, \ \exists N \in \mathbb{N} \ \text{s.t.} \ \forall n \geq N, \ |a_n - s| < \varepsilon. \tag{6.15}$$

Let $\varepsilon > 0$. Since s is the smallest upper bound for A, this means

$$s - \varepsilon$$

is not an upper bound. Therefore there exists $N \in \mathbb{N}$ such that

$$s - \varepsilon < a_N . \tag{6.16}$$

Let $n \geq N$. Since a_n is increasing, we have

$$a_N \leq a_n , \quad \forall \, n \geq N . \tag{6.17}$$

Moreover s is the supremum of A, so that

$$a_n \leq s < s + \varepsilon , \quad \forall \, n \in \mathbb{N} . \tag{6.18}$$

Putting together estimates (6.16)-(6.17)-(6.18) we get

$$s - \varepsilon < a_N \leq a_n \leq s < s + \varepsilon , \quad \forall \, n \geq N .$$

The above implies

$$s - \varepsilon < a_n < s + \varepsilon , \quad \forall \, n \geq N ,$$

which is equivalent to (6.15).

2. (a_n) is decreasing: With a similar proof, one can show that

$$\lim_{n \to \infty} a_n = i .$$

This is left as an exercise.

6.9.1 Example: Euler's Number

As an application of the Monotone Convergence Theorem we can give a formal definition for the Euler's Number

$$e = 2.71828182845904523536 \ldots$$

Theorem 6.51

Consider the sequence

$$a_n = \left(1 + \frac{1}{n} \right)^n .$$

We have that:

1. (a_n) is monotone increasing,
2. (a_n) is bounded.

In particular (a_n) is convergent.

Proof

Part 1. We prove that (a_n) is increasing

$$a_n \geq a_{n-1}, \quad \forall\, n \in \mathbb{N},$$

which by definition is equivalent to

$$\left(1 + \frac{1}{n}\right)^n \geq \left(1 + \frac{1}{n-1}\right)^{n-1}, \quad \forall\, n \in \mathbb{N}.$$

Summing the fractions we get

$$\left(\frac{n+1}{n}\right)^n \geq \left(\frac{n}{n-1}\right)^{n-1}.$$

Multiplying by $((n-1)/n)^n$ we obtain

$$\left(\frac{n-1}{n}\right)^n \left(\frac{n+1}{n}\right)^n \geq \frac{n-1}{n},$$

which simplifies to

$$\left(1 - \frac{1}{n^2}\right)^n \geq 1 - \frac{1}{n}, \quad \forall\, n \in \mathbb{N}. \tag{6.19}$$

Therefore (a_n) is increasing if and only if (6.19) holds. Recall Bernoulli's inequality from Lemma 6.41: For $x \in \mathbb{R}$, $x > -1$, it holds

$$(1 + x)^n \geq 1 + nx, \quad \forall\, n \in \mathbb{N}.$$

Appliying Bernoulli's inequality with

$$x = -\frac{1}{n^2}$$

yields

$$\left(1 - \frac{1}{n^2}\right)^n \geq 1 + n\left(-\frac{1}{n^2}\right) = 1 - \frac{1}{n},$$

which is exactly (6.19). Then (a_n) is increasing.

Part 2. We have to prove that (a_n) is bounded, that is, that there exists $M > 0$ such that

$$|a_n| \leq M, \quad \forall\, n \in \mathbb{N}.$$

To this end, introduce the sequence (b_n) by setting

$$b_n := \left(1 + \frac{1}{n}\right)^{n+1}.$$

The sequence (b_n) is decreasing.

To prove (b_n) is decreasing, we need to show that

$$b_{n-1} \geq b_n, \quad \forall\, n \in \mathbb{N}.$$

By definition of b_n, the above reads

$$\left(1 + \frac{1}{n-1}\right)^n \geq \left(1 + \frac{1}{n}\right)^{n+1}, \quad \forall\, n \in \mathbb{N}.$$

Summing the terms inside the brackets, the above is equivalent to

$$\left(\frac{n}{n-1}\right)^n \geq \left(\frac{n+1}{n}\right)^n \left(\frac{n+1}{n}\right).$$

Multiplying by $(n/(n+1))^n$ we get

$$\left(\frac{n^2}{n^2-1}\right)^n \geq \left(\frac{n+1}{n}\right).$$

The above is equivalent to

$$\left(1 + \frac{1}{n^2-1}\right)^n \geq \left(1 + \frac{1}{n}\right). \tag{6.20}$$

Therefore (b_n) is decreasing if and only if (6.20) holds for all $n \in \mathbb{N}$. By choosing

$$x = \frac{1}{n^2-1}$$

in Bernoulli's inequality, we obtain

$$\left(1 + \frac{1}{n^2-1}\right)^n \geq 1 + n\left(\frac{1}{n^2-1}\right)$$
$$= 1 + \frac{n}{n^2-1}$$
$$\geq 1 + \frac{1}{n},$$

where in the last inequality we used that

$$\frac{n}{n^2-1} > \frac{1}{n},$$

which holds, being equivalent to $n^2 > n^2 - 1$. We have therefore proven (6.20), and hence (b_n) is decreasing.

We now observe that For all $n \in \mathbb{N}$

$$b_n = \left(1 + \frac{1}{n}\right)^{n+1}$$
$$= \left(1 + \frac{1}{n}\right)^n \left(1 + \frac{1}{n}\right)$$
$$= a_n \left(1 + \frac{1}{n}\right)$$
$$> a_n \,.$$

Since (a_n) is increasing and (b_n) is decreasing, in particular

$$a_n \geq a_1 \,, \quad b_n \leq b_1 \,.$$

Therefore

$$a_1 \leq a_n < b_n \leq b_1 \,, \quad \forall\, n \in \mathbb{N} \,.$$

We compute

$$a_1 = 2 \,, \quad b_1 = 4 \,,$$

from which we get

$$2 \leq a_n \leq 4 \,, \quad \forall\, n \in \mathbb{N} \,.$$

Therefore

$$|a_n| \leq 4 \,, \quad \forall\, n \in \mathbb{N} \,,$$

showing that (a_n) is bounded.

Part 3. The sequence (a_n) is increasing and bounded above. Therefore (a_n) is convergent by the Monotone Convergence Theorem 6.50.

Thanks to Theorem 6.51 we can define the Euler's Number e.

Definition 6.52: Euler's Number

The Euler's number is defined as

$$e := \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \,.$$

Setting $n = 1000$ in the formula for (a_n), we get an approximation of e:

$$e \approx a_{1000} = 2.7169 \,.$$

6.10 Some important limits

In this section we investigate limits of some sequences to which the Limit Tests do not apply.

Theorem 6.53

Let $x \in \mathbb{R}$, with $x > 0$. Then

$$\lim_{n \to \infty} \sqrt[n]{x} = 1 \,.$$

Proof

Step 1. Assume $x \geq 1$. In this case

$$\sqrt[n]{x} \geq 1 \,.$$

Define

$$b_n := \sqrt[n]{x} - 1 \,,$$

so that $b_n \geq 0$. By Bernoulli's Inequality we have

$$x = (1 + b_n)^n \geq 1 + nb_n \,.$$

Therefore

$$0 \leq b_n \leq \frac{x - 1}{n} \,.$$

Since

$$\frac{x - 1}{n} \longrightarrow 0 \,,$$

by the Squeeze Theorem we infer $b_n \to 0$, and hence

$$\sqrt[n]{x} = 1 + b_n \longrightarrow 1 + 0 = 1 \,,$$

by the Algebra of Limits.
Step 2. Assume $0 < x < 1$. In this case

$$\frac{1}{x} > 1 \,.$$

Therefore

$$\lim_{n \to \infty} \sqrt[n]{1/x} = 1 \,.$$

by Step 1. Therefore

$$\sqrt[n]{x} = \frac{1}{\sqrt[n]{1/x}} \longrightarrow \frac{1}{1} = 1 \,,$$

by the Algebra of Limits.

Theorem 6.54

Let (a_n) be a sequence such that $a_n \to 0$. Then

$$\sin(a_n) \to 0, \quad \cos(a_n) \to 1.$$

Proof

Assume that $a_n \to 0$ and set

$$\varepsilon := \frac{\pi}{2}.$$

By the convergence $a_m \to 0$ there exists $N \in \mathbb{N}$ such that

$$|a_n| < \varepsilon = \frac{\pi}{2} \quad \forall\, n \geq N. \tag{6.21}$$

Step 1. We prove that

$$\sin(a_n) \to 0.$$

By elementary trigonometry we have

$$0 \leq |\sin(x)| = \sin|x| \leq |x|, \quad \forall\, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Therefore, since (6.21) holds, we can substitute $x = a_n$ in the above inequality to get

$$0 \leq |\sin(a_n)| \leq |a_n|, \quad \forall\, n \geq \mathbb{N}.$$

Since $a_n \to 0$, we also have $|a_n| \to 0$. Therefore $|\sin(a_n)| \to 0$ by the Squeeze Theorem. This immediately implies $\sin(a_n) \to 0$.
Step 2. We prove that

$$\cos(a_n) \to 1.$$

Inverting the relation

$$\cos^2(x) + \sin^2(x) = 1,$$

we obtain

$$\cos(x) = \pm\sqrt{1 - \sin^2(x)}.$$

We have that $\cos(x) \geq 0$ for $-\pi/2 \leq x \leq \pi/2$. Thus

$$\cos(x) = \sqrt{1 - \sin^2(x)}, \quad \forall\, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

Since (6.21) holds, we can set $x = a_n$ in the above inequality and obtain

$$\cos(a_n) = \sqrt{1 - \sin^2(a_n)}, \quad \forall\, n \geq N.$$

By Step 1 we know that $\sin(a_n) \to 0$. Therefore, by the Algebra of Limits,

$$1 - \sin^2(a_n) \longrightarrow 1 - 0 \cdot 0 = 1.$$

Using Theorem 6.32 we have

$$\cos(a_n) = \sqrt{1 - \sin^2(a_n)} \longrightarrow \sqrt{1} = 1,$$

concluding the proof.

Theorem 6.55

Suppose (a_n) is such that $a_n \to 0$ and

$$a_n \neq 0, \quad \forall\, n \in \mathbb{N}.$$

Then

$$\lim_{n \to \infty} \frac{\sin(a_n)}{a_n} = 1.$$

Proof

The following elementary trigonometric inequality holds:

$$\sin(x) < x < \tan(x), \quad \forall\, x \in \left[0, \frac{\pi}{2}\right].$$

Note that $\sin x > 0$ for $0 < x < \pi/2$. Therefore we can divide the above inequality by $\sin(x)$ and take the reciprocals to get

$$\cos(x) < \frac{\sin(x)}{x} < 1, \quad \forall\, x \in \left(0, \frac{\pi}{2}\right].$$

If $-\pi/2 < x < 0$, we can apply the above inequality to $-x$ to obtain

$$\cos(-x) < \frac{\sin(-x)}{-x} < 1.$$

Recalling that $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$, we get

$$\cos(x) < \frac{\sin(x)}{x} < 1, \quad \forall\, x \in \left(-\frac{\pi}{2}, 0\right].$$

Thus

$$\cos(x) < \frac{\sin(x)}{x} < 1, \quad \forall\, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \setminus \{0\}. \tag{6.22}$$

Let

$$\varepsilon := \frac{\pi}{2}.$$

Since $a_n \to 0$, there exists $N \in \mathbb{N}$ such that

$$|a_n| < \varepsilon = \frac{\pi}{2}, \quad \forall\, n \geq N.$$

Since $a_n \neq 0$ by assumption, the above shows that

$$a_n \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \setminus \{0\}, \quad \forall\, n \geq \mathbb{N}.$$

Therefore we can substitute $x = a_n$ into (6.22) to get

$$\cos(a_n) < \frac{\sin(a_n)}{a_n} < 1, \quad \forall\, n \geq N.$$

We have

$$\cos(a_n) \to 1$$

by Theorem 6.54. By the Squeeze Theorem we conclude that

$$\lim_{n \to \infty} \frac{\sin(a_n)}{a_n} = 1.$$

Warning

You might be tempted to apply L'Hôpital's rule (which we did not cover in these Lecture Notes) to compute

$$\lim_{x \to 0} \frac{\sin(x)}{x}.$$

This would yield the correct limit

$$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{(\sin(x))'}{(x)'} = \lim_{x \to 0} \cos(x) = 1.$$

However this is a circular argument, since the derivative of $\sin(x)$ at $x = 0$ is defined as the limit

$$\lim_{x \to 0} \frac{\sin(x)}{x}.$$

Theorem 6.56

Suppose (a_n) is such that $a_n \to 0$ and

$$a_n \neq 0, \quad \forall\, n \in \mathbb{N}.$$

Then

$$\lim_{n \to \infty} \frac{1 - \cos(a_n)}{(a_n)^2} = \frac{1}{2}, \quad \lim_{n \to \infty} \frac{1 - \cos(a_n)}{a_n} = 0.$$

Proof

Step 1. By Theorem 6.54 and Theorem 6.55, we have

$$\cos(a_n) \to 1, \quad \frac{\sin(a_n)}{a_n} \to 1.$$

Therefore

$$
\begin{aligned}
\frac{1 - \cos(a_n)}{(a_n)^2} &= \frac{1 - \cos(a_n)}{(a_n)^2} \frac{1 + \cos(a_n)}{1 + \cos(a_n)} \\
&= \frac{1 - \cos^2(a_n)}{(a_n)^2} \frac{1}{1 + \cos(a_n)} \\
&= \left(\frac{\sin(a_n)}{a_n} \right)^2 \frac{1}{1 + \cos(a_n)} \longrightarrow 1 \cdot \frac{1}{1 + 1} = \frac{1}{2},
\end{aligned}
$$

where in the last line we use the Algebra of Limits.
Step 2. We have

$$\frac{1 - \cos(a_n)}{a_n} = a_n \cdot \frac{1 - \cos(a_n)}{(a_n)^2} \longrightarrow 0 \cdot \frac{1}{2} = 0,$$

using Step 1 and the Algebra of Limits.

Example 6.57

- We have

$$\lim_{n \to \infty} n \sin\left(\frac{1}{n} \right) = 1. \tag{6.23}$$

This is because

$$n \sin\left(\frac{1}{n} \right) = \frac{\sin\left(\frac{1}{n} \right)}{\frac{1}{n}} \longrightarrow 1,$$

by Theorem 6.55 with $a_n = 1/n$.

- We have
$$\lim_{n\to\infty} n^2 \left(1 - \cos\left(\frac{1}{n}\right)\right) = \frac{1}{2}.$$ (6.24)

Indeed,
$$n^2 \left(1 - \cos\left(\frac{1}{n}\right)\right) = \frac{1 - \cos\left(\frac{1}{n}\right)}{\frac{1}{n^2}} \longrightarrow \frac{1}{2},$$

by applying Theorem 6.56 with $a_n = 1/n$.

- We have
$$\lim_{n\to\infty} \frac{n\left(1 - \cos\left(\frac{1}{n}\right)\right)}{\sin\left(\frac{1}{n}\right)} = \frac{1}{2}.$$

Indeed, using (6.24)-(6.23) and the Algebra of Limits
$$\frac{n\left(1 - \cos\left(\frac{1}{n}\right)\right)}{\sin\left(\frac{1}{n}\right)} = \frac{n^2 \left(1 - \cos\left(\frac{1}{n}\right)\right)}{n \sin\left(\frac{1}{n}\right)} \longrightarrow \frac{1/2}{1} = \frac{1}{2}.$$

- We have
$$\lim_{n\to\infty} n \cos\left(\frac{2}{n}\right) \sin\left(\frac{2}{n}\right) = 2.$$

This is because
$$\cos\left(\frac{2}{n}\right) \longrightarrow 1,$$

by Theorem 6.54 applied with $a_n = 2/n$. Moreover
$$\frac{\sin\left(\frac{2}{n}\right)}{\frac{2}{n}} \longrightarrow 1,$$

by Theorem 6.55 applied with $a_n = 2/n$. Therefore
$$n \cos\left(\frac{2}{n}\right) \sin\left(\frac{2}{n}\right) = 2 \cdot \cos\left(\frac{2}{n}\right) \cdot \frac{\sin\left(\frac{2}{n}\right)}{\frac{2}{n}} \longrightarrow 2 \cdot 1 \cdot 1 = 2$$

where we used the Algebra of Limits.

- We have
$$\lim_{n \to \infty} \frac{n^2 + 1}{n + 1} \sin\left(\frac{1}{n}\right) = 1 \,.$$

To prove it, note that

$$\frac{n^2 + 1}{n + 1} \sin\left(\frac{1}{n}\right) = \left(\frac{1 + \dfrac{1}{n^2}}{1 + \dfrac{1}{n}}\right) \cdot \left(n \sin\left(\frac{1}{n}\right)\right) \longrightarrow \frac{1 + 0}{1 + 0} \cdot 1 = 1 \,,$$

where we used (6.23) and the Algebra of Limits.

7 Sequences in $\mathbb{C}$

The theory for sequences in $\mathbb{C}$ is very similar to that of sequences in $\mathbb{R}$. In $\mathbb{R}$, we said that a sequence (a_n) converges to some number $a \in \mathbb{R}$ if for all $\varepsilon > 0$, it holds

$$|a_n - a| < \varepsilon$$

for all n suffieciently large. The definition of convergence in $\mathbb{C}$ is essentially the same, with the absolute value replaced by the complex modulus.

7.1 Definition and convergence

First of all, let us give the formal definition of sequence in $\mathbb{C}$.

Definition 7.1: Sequence of Complex numbers

A sequence a in $\mathbb{C}$ is a function

$$a : \mathbb{N} \to \mathbb{C}.$$

For $n \in \mathbb{N}$, we denote the n-th element of the sequence a by

$$a_n = a(n)$$

and write the sequence as

$$(a_n)_{n \in \mathbb{N}} \quad \text{or} \quad (a_n).$$

In the following we define convergent sequences in $\mathbb{C}$.

Definition 7.2: Convergent sequence in $\mathbb{C}$

We say that a sequence $(a_n)_{n \in \mathbb{N}}$ in $\mathbb{C}$ **converges** to $a \in \mathbb{C}$, or equivalently has limit a, denoted by

$$\lim_{n \to \infty} a_n = a \quad \text{or} \quad a_n \to a,$$

if for all $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}, n \geq N$ it holds that

$$|a_n - a| < \varepsilon.$$

Using quantifiers, we can write this as

$$\forall \varepsilon > 0, \; \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N, \; |a_n - a| < \varepsilon.$$

If there exists $a \in \mathbb{C}$ such that $\lim_{n \to \infty} a_n = a$, then we say that the sequence $(a_n)_{n \in \mathbb{N}}$ is **convergent**.

Important

In Definition 7.2 we still take ε to be real. This makes sense, since

$$|z| = \sqrt{x^2 + y^2} \in \mathbb{R}$$

for all $z = x + iy \in \mathbb{C}$.

Example 7.3

Using Definition 7.2, prove that

$$\lim_{n \to \infty} \frac{(3+i)n - 7i}{n} = 3 + i.$$

Part 1. Rough Work. Let $\varepsilon > 0$. We would like to understand for which values of n the following holds:

$$\left| \frac{(3+i)n - 7i}{n} - (3+i) \right| < \varepsilon.$$

We have

$$\left| \frac{(3+i)n - 7i}{n} - (3+i) \right| = \left| \frac{(3+i)n - 7i - (3+i)n}{n} \right|$$

$$= \frac{|-7i|}{n}$$

$$= \frac{7}{n}.$$

Therefore

$$\frac{7}{n} < \varepsilon \quad \Longleftrightarrow \quad n > \frac{7}{\varepsilon}.$$

Part 2. Formal Proof. We want to prove that

$$\forall \varepsilon > 0, \; \exists N \in \mathbb{N} \text{ s.t. } \forall n \geq N, \; \left| \frac{(3+i)n - 7i}{n} - (3+i) \right| < \varepsilon.$$

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that

$$N > \frac{7}{\varepsilon}\,.$$

The above is equivalent to

$$\frac{7}{N} < \varepsilon\,.$$

For $n \geq N$ we have

$$\left| \frac{(3+i)n - 7i}{n} - (3+i) \right| = \frac{7}{n} \leq \frac{7}{N} < \varepsilon\,.$$

7.2 Boundedness

Boundedness plays an important role for complex sequences.

Definition 7.4: Bounded sequence in $\mathbb{C}$

A sequence (a_n) in $\mathbb{C}$ is called **bounded** if there exists a constant $M \in \mathbb{R}$, with $M > 0$, such that

$$|a_n| \leq M\,, \quad \forall\, n \in \mathbb{N}\,.$$

As it happens in $\mathbb{R}$, we have that complex sequences which converge are also bounded.

Theorem 7.5

If a sequence (a_n) in $\mathbb{C}$ converges, then the sequence is bounded.

The proof is identical to the one in $\mathbb{R}$, and is hence omitted. Similarly to real sequences, we can define divergent complex sequences.

Definition 7.6: Divergent sequences in $\mathbb{C}$

We say that a sequence (a_n) in $\mathbb{C}$ is **divergent** if it is not convergent.

As a corollary of Theorem 7.5 we have the following.

Corollary 7.7

Let (a_n) be a complex sequence. If (a_n) is not bounded, then it is divergent.

7.3 Algebra of limits in $\mathbb{C}$

Most of the results about limits that we have shown in $\mathbb{R}$ also hold in $\mathbb{C}$. The first result is the Algebra of Limits.

Theorem 7.8: Algebra of limits in $\mathbb{C}$

Let (a_n) and (b_n) be sequences in $\mathbb{C}$. Suppose that

$$\lim_{n\to\infty} a_n = a, \quad \lim_{n\to\infty} b_n = b,$$

for some $a, b \in \mathbb{C}$. Then,

1. Limit of sum is the sum of limits:
$$\lim_{n\to\infty} (a_n \pm b_n) = a \pm b$$

2. Limit of product is the product of limits:
$$\lim_{n\to\infty} (a_n b_n) = ab$$

3. If $b_n \neq 0$ for all $n \in \mathbb{N}$ and $b \neq 0$, then

$$\lim_{n\to\infty} \left(\frac{a_n}{b_n} \right) = \frac{a}{b}$$

The proof of Theorem 7.8 follows word by word the proof of the Algebra of Limits for sequences in $\mathbb{R}$: one just needs to replace the absolute value by the complex modulus.

We can use the Algebra of Limits to compute limits of complex sequences.

Example 7.9

Let (a_n) be the sequence defined by

$$a_n := \frac{(2-i)n^2 + 6in - 5 - 3i}{(6+3i)n^2 + 11i}.$$

The largest power of n in the denominator is n^2. We hence divide by n^2 to obtain

$$a_n = \frac{(2-i)n^2 + 6in - 5 - 3i}{(6+3i)n^2 + 11i} = \frac{(2-i) + \dfrac{6i}{n} - \dfrac{5}{n^2} - \dfrac{3i}{n^2}}{(6+3i) + \dfrac{11i}{n^2}}$$

By the Algebra of Limits the right hand side converges to

$$\frac{(2-i) + 0 - 0 - 0}{(6+3i) + 0} = \frac{2-i}{6+3i}$$

By performing the complex division, we can write the limit in the form $x + iy$:

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \frac{2-i}{6+3i} \\
&= \frac{(2-i)(6-3i)}{(6+3i)(6-3i)} \\
&= \frac{12 - 6i - 6i + 3i^2}{36 - 9i^2} \\
&= \frac{9 - 12i}{45} \\
&= \frac{1}{5} - \frac{4}{15}i
\end{aligned}$$

7.4 Convergence to zero

One of the results that cannot hold for complex sequences is the Squeeze Theorem. Indeed the chain of inequalities

$$b_n \leq a_n \leq c_n$$

would not make sense in $\mathbb{C}$, since there is no order relation.

We can however prove the following (weaker) result.

Theorem 7.10

Let (a_n) be a sequence in $\mathbb{C}$ and suppose that

$$\lim_{n\to\infty} |a_n| = 0 \,.$$

Then

$$\lim_{n \to \infty} a_n = 0 \,.$$

Proof

Assume that $|a_n| \to 0$. We need to show that

$$\forall \, \varepsilon > 0 \,, \, \exists \, N \in \mathbb{N} \text{ s.t. } \forall \, n \geq N \,, \, |a_n - 0| < \varepsilon \,.$$

Let $\varepsilon > 0$. Since $|a_n| \to 0$, there exists $N \in \mathbb{N}$ such that

$$\big||a_n| - 0\big| < \varepsilon \,, \quad \forall \, n \geq N \,.$$

Let $n \geq N$. Then,

$$
\begin{aligned}
|a_n - 0| &= |a_n| \\
&= |a_n| - 0 \\
&= \big||a_n| - 0\big| \\
&< \varepsilon \,.
\end{aligned}
$$

Note that the sequence $|a_n|$ is real. Therefore the convergence of $|a_n|$ can be studied using convergence results in $\mathbb{R}$.

Example 7.11

Consider the complex sequence

$$a_n = \left(\frac{1}{2} + \frac{1}{3}i \right)^n \,.$$

Prove that $a_n \to 0$.

Proof. We have

$$
\begin{aligned}
|a_n| &= \left| \left(\frac{1}{2} + \frac{1}{3}i \right)^n \right| \\
&= \left| \frac{1}{2} + \frac{1}{3}i \right|^n \\
&= \left(\sqrt{ \left(\frac{1}{2} \right)^2 + \left(\frac{1}{3} \right)^2 } \right)^n \\
&= \left(\sqrt{ \frac{13}{36} } \right)^n \,.
\end{aligned}
$$

Since
$$\left| \sqrt{\frac{13}{36}} \right| < 1 \,,$$

by the Geometric Sequence Test for real sequences, we conclude that

$$|a_n| \to 0 \,.$$

Hence $a_n \to 0$ by Theorem 7.10.

Although the Squeeze Theorem cannot be used for complex sequences, sometimes it can be used to deal with real terms in a complex sequence.

Example 7.12

Consider the sequence
$$a_n := \frac{2i \cos(3n)n + (7 - i)n^2}{3n^2 + 2in + \sin(2n)} \,.$$

Prove that
$$\lim_{n \to \infty} a_n = \frac{7}{3} - \frac{1}{3}i \,.$$

Proof. We divide by the largest power in the denominator, to get

$$a_n = \frac{\dfrac{2i \cos(3n)}{n} + (7 - i)}{3 + \dfrac{2i}{n} + \dfrac{\sin(2n)}{n^2}} \,.$$

Notice that
$$-1 \leq \cos(3n) \leq 1 \,, \quad \forall n \in \mathbb{N} \,,$$

and thus
$$-\frac{2}{n} \leq \frac{2 \cos(3n)}{n} \leq \frac{2}{n} \,, \quad \forall n \in \mathbb{N} \,.$$

Since
$$-\frac{2}{n} \longrightarrow 0 \,, \quad \frac{2}{n} \longrightarrow 0 \,,$$

by the Squeeze Theorem we conclude that also

$$\frac{2 \cos(3n)}{n} \to 0 \,.$$

In particular we have shown that

$$\left| \frac{2i \cos(3n)}{n} \right| = \left| \frac{2 \cos(3n)}{n} \right| \to 0 \,.$$

Using Theorem 7.10 we infer

$$\frac{2i\cos(3n)}{n} \to 0 \,.$$

Similarly,

$$-\frac{1}{n^2} \leq \frac{\sin(2n)}{n^2} \leq -\frac{1}{n^2} \,, \quad \forall\, n \in \mathbb{N} \,.$$

Since

$$-\frac{1}{n^2} \longrightarrow 0 \,, \quad \frac{1}{n^2} \longrightarrow 0 \,,$$

by the Squeeze Theorem we conclude

$$\frac{\sin(2n)}{n^2} \longrightarrow 0 \,.$$

Finally, we have

$$\left|\frac{2i}{n}\right| = \frac{2}{n} \longrightarrow 0 \,,$$

and therefore

$$\frac{2i}{n} \longrightarrow 0$$

by Theorem 7.10. Using the Algebra of Limits in $\mathbb{C}$ we conclude

$$a_n = \frac{\dfrac{2i\cos(3n)}{n} + (7-i)}{3 + \dfrac{2i}{n} + \dfrac{\sin(2n)}{n^2}} \longrightarrow \frac{0 + (7-i)}{3 + 0 + 0} = \frac{7}{3} - \frac{1}{3}i \,.$$

7.5 Geometric sequence Test and Ratio Test in $\mathbb{C}$

The Geometric Sequence Test and Ratio Test can be generalized to complex sequences.

Theorem 7.13:   Geometric sequence Test in $\mathbb{C}$

Let $x \in \mathbb{C}$ and let $(a_n)_{n\in\mathbb{N}}$ be the geometric sequence in $\mathbb{C}$ defined by

$$a_n := x^n \,.$$

We have:

1. If $|x| < 1$, then
$$\lim_{n\to\infty} a_n = 0 \,.$$

2. If $|x| > 1$, then sequence (a_n) is unbounded, and hence divergent.

The proof can be obtained as in the real case, replacing the absolute value by the modulus.

Example 7.14

- Let

$$a_n = \frac{(-1 + 4i)^n}{(7 + 3i)^n} \, .$$

We first rewrite

$$a_n = \frac{(-1 + 4i)^n}{(7 + 3i)^n} = \left(\frac{-1 + 4i}{7 + 3i} \right)^n$$

Then, we compute

$$\left| \frac{-1 + 4i}{7 + 3i} \right| = \frac{|-1 + 4i|}{|7 + 3i|}$$

$$= \frac{\sqrt{(-1)^2 + 4^2}}{\sqrt{7^2 + 3^2}}$$

$$= \frac{\sqrt{17}}{\sqrt{58}}$$

$$= \sqrt{\frac{17}{58}}$$

$$< 1$$

By the Geometric Sequence Test $a_n \to 0$.

- Let

$$b_n = \frac{(-5 + 12i)^n}{(3 - 4i)^n} \, .$$

We first rewrite

$$b_n = \frac{(-5 + 12i)^n}{(3 - 4i)^n} = \left(\frac{-5 + 12i}{3 - 4i} \right)^n \, .$$

We compute

$$\left| \frac{-5 + 12i}{3 - 4i} \right| = \frac{|-5 + 12i|}{|3 - 4i|}$$

$$= \frac{\sqrt{5^2 + (-12)^2}}{\sqrt{3^2 + (-4)^2}}$$

$$= \frac{13}{5}$$

$$> 1 \, .$$

By the Geometric Sequence Test, the sequence (b_n) does not converge.

- Let
$$c_n = e^{\frac{i\pi}{2}n}.$$

We have
$$|c_n| = \left| e^{\frac{i\pi}{2}n} \right| = 1,$$

and hence the Geometric Sequence Test cannot be applied. However, we can see that
$$c_n = (i, -1, -i, 1, i, -1, -i, 1, \ldots),$$

that is, c_n assumes only the values $\{i, -1, -i, 1\}$, and each of them is assumed infinitely many times. Thus c_n is oscillating and it is divergent.

We now provide the statement of the Ratio Test in $\mathbb{C}$.

Theorem 7.15: Ratio Test in $\mathbb{C}$

Let (a_n) be a sequence in $\mathbb{C}$ such that
$$a_n \neq 0, \quad \forall\, n \in \mathbb{N}.$$

1. Suppose that the following limit exists:
$$L := \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

   Then,

   - If $L < 1$ we have
   $$\lim_{n \to \infty} a_n = 0.$$

   - If $L > 1$, the sequence (a_n) is unbounded, and hence does not converge.

2. Suppose that there exists $N \in N$ and $L > 1$ such that
$$\left| \frac{a_{n+1}}{a_n} \right| \geq L, \quad \forall\, n \geq N.$$

   Then the sequence (a_n) is unbounded, and hence does not converge.

The proof of Theorem 7.15 follows word by word the proof of the Ratio Test Theorem in $\mathbb{R}$, and only two minor modifications are needed:

- Replace the absolute value with the complex modulus,
- Instead of the Squeeze Theorem, use Theorem 7.10.

> **Example 7.16**
>
> Let
> $$a_n = \frac{(4 - 3i)^n}{(2n)!} \, .$$
>
> Prove that $a_n \to 0$.
>
> *Proof.* We compute
> $$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(4 - 3i)^{n+1}}{(2(n+1))!} \frac{(2n)!}{(4 - 3i)^n} \right|$$
> $$= \frac{|4 - 3i|^{n+1}}{|4 - 3i|^n} \cdot \frac{(2n)!}{(2n+2)!}$$
> $$= \frac{|4 - 3i|}{(2n+2)(2n+1)}$$
> $$= \frac{\sqrt{4^2 + (-3)^2}}{(2n+2)(2n+1)}$$
> $$= \frac{5}{(2n+2)(2n+1)}$$
> $$= \frac{\dfrac{5}{n^2}}{\left(2 + \dfrac{2}{n}\right)\left(2 + \dfrac{1}{n}\right)} \longrightarrow L = 0 \, .$$
>
> Since $L = 0 < 1$, by the Ratio Test in $\mathbb{C}$ we infer $a_n \to 0$.

7.6 Convergence of real and imaginary part

A complex number $z \in \mathbb{C}$ can be written as
$$z = a + bi$$
for some $a, b \in \mathbb{R}$, where

- a is the real part of z,
- b the imaginary part of z.

We can prove that a complex sequence converges if and only if both the real parts and the imaginary parts converge.

Theorem 7.17

Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}$. For $n \in \mathbb{N}$, let $a_n, b_n \in \mathbb{R}$ such that

$$z_n = a_n + b_n i\,.$$

Let

$$z = a + bi$$

with $a, b \in \mathbb{R}$. Then

$$\lim_{n\to\infty} z_n = z \quad \Longleftrightarrow \quad \lim_{n\to\infty} a_n = a\,, \quad \lim_{n\to\infty} b_n = b.$$

Proof

Part 1. Suppose that

$$\lim_{n\to\infty} z_n = z\,.$$

To prove that $a_n \to a$ we need to show that

$$\forall\, \varepsilon > 0\,, \exists\, N \in \mathbb{N} \text{ s.t. } \forall\, n \geq N\,, \; |a_n - a| < \varepsilon\,.$$

Let $\varepsilon > 0$. Since $z_n \to z$, there exists $N \in \mathbb{N}$ such that

$$|z_n - z| < \varepsilon\,, \quad \forall\, n \geq N\,.$$

Let $n \geq N$. Then

$$\begin{aligned}
|a_n - a| &= \sqrt{(a_n - a)^2} \\
&\leq \sqrt{(a_n - a)^2 + (b_n - b)^2} \\
&= |(a_n - a) + (b_n - b)\,i| \\
&= |(a_n + b_n i) - (a + bi)| \\
&= |z_n - z| \\
&< \varepsilon\,.
\end{aligned}$$

The proof for $b_n \to b$ is similar.
Part 2. Suppose that

$$\lim_{n\to\infty} a_n = a\,, \quad \lim_{n\to\infty} b_n = b\,.$$

To prove that $z_n \to z$ we need to show that

$$\forall\, \varepsilon > 0\,, \exists\, N \in \mathbb{N} \text{ s.t. } \forall\, n \geq N\,, \; |z_n - z| < \varepsilon\,.$$

Let $\varepsilon > 0$. Since $a_n \to a$, there exists $N_1 \in \mathbb{N}$ such that

$$|a_n - a| < \frac{\varepsilon}{2}\,, \quad \forall\, n \geq N_1\,.$$

Since $b_n \to b$, there exists $N_2 \in \mathbb{N}$ such that

$$|b_n - b| < \frac{\varepsilon}{2}, \quad \forall\, n \geq N_2\,.$$

Let

$$N := \max\{N_1, N_2\}\,.$$

Let $n \geq N$. By the triangle inequality,

$$
\begin{aligned}
|z_n - z| &= |(a_n + b_n i) - (a + bi)| \\
&= |(a_n - a) + (b_n - b)\, i| \\
&\leq |a_n - a| + |b_n - b| \cdot |i| \\
&= |a_n - a| + |b_n - b| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon\,.
\end{aligned}
$$

Example 7.18

Consider the complex sequence

$$z_n := \frac{\left(4n + 3n^2 i\right)\left(2n^2 + i\right)}{5n^4}$$

Show that

$$\lim_{n \to \infty} z_n = \frac{6}{5}i\,.$$

Proof. Let us find the real and imaginary parts of z_n

$$
\begin{aligned}
z_n &= \frac{\left(4n + 3n^2 i\right)\left(2n^2 + i\right)}{5n^4} \\
&= \frac{8n^3 + 4ni + 6n^4 i + 3n^2 i^2}{5n^4} \\
&= \frac{8n^3 - 3n^2}{5n^4} + \frac{6n^4 + 4n}{5n^4}i \\
&= a_n + b_n i\,.
\end{aligned}
$$

Using the Algebra of Limits for real sequences we have that

$$a_n = \frac{8n^3 - 3n^2}{5n^4} = \frac{\dfrac{8}{n} - \dfrac{3}{n^2}}{5} \longrightarrow \frac{0 - 0}{5} = 0\,,$$

and

$$b_n = \frac{6n^4 + 4n}{5n^4} = \frac{6 + \dfrac{4}{n^3}}{5} \longrightarrow \frac{6 + 0}{5} = \frac{6}{5}\,.$$

By Theorem 7.17 we conclude

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} a_n + i \lim_{n\to\infty} b_n = 0 + \frac{6}{5}i = \frac{6}{5}i \,.$$

8 Series

A series is the sum of all terms in a sequence:

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$

Since we are dealing with an infinite amount of terms, we need to be careful. For example, consider the series

$$\sum_{n=1}^{\infty} (-1)^n = -1 + 1 - 1 + 1 - 1 + 1 - \ldots \tag{8.1}$$

If we sum the terms in pairs, we obtain

$$\sum_{n=1}^{\infty} (-1)^n = (-1 + 1) + (-1 + 1) + (-1 + 1) + \ldots = 0 \, .$$

If we reorder the terms, we obtain a different result

$$\begin{aligned}
\sum_{k=1}^{\infty} (-1)^n &= (1 + 1) - 1 + (1 + 1) - 1 + \ldots \\
&= (2 - 1) + (2 - 1) + \ldots \\
&= 1 + 1 + \ldots \\
&= \infty \, .
\end{aligned}$$

Therefore commutativity of the sum does not hold when summing infinitely many terms. We need a good definition of convergence.

8.1 Convergent series

We will develop the theory of series for sequences in $\mathbb{C}$. All the results for complex series will also hold for series in $\mathbb{R}$. Some results will only hold for series in $\mathbb{R}$: this will be the case for comparison tests in which the order relation of $\mathbb{R}$ is needed.

We start by defining partial sums.

Definition 8.1: Partial sums

Let (a_n) be a sequence in $\mathbb{C}$. The k-th partial sum of (a_n) is

$$s_k := a_1 + a_2 + \dots + a_k = \sum_{n=1}^{k} a_n$$

This sequence $(s_k)_{k \in \mathbb{N}}$ is called the sequence of **partial sums**.

We can use the sequence of partial sums to define convergence of a series.

Definition 8.2: Convergent series

Let (a_n) be a sequence in $\mathbb{C}$. We denote the series of $(a_n)_{n \in \mathbb{N}}$ by

$$\sum_{n=1}^{\infty} a_n$$

We say that this series **converges** to $s \in \mathbb{C}$ if

$$\lim_{k \to \infty} \sum_{n=1}^{k} a_n = \lim_{k \to \infty} s_k = s \,.$$

In this case we write

$$\sum_{n=1}^{\infty} a_n = s \,.$$

Definition 8.3: Divergent series

Let (a_n) be a sequence in $\mathbb{C}$. The series

$$\sum_{n=1}^{\infty} a_n$$

is **divergent** if the sequence of partial sums (s_k) is divergent.

Example 8.4

Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}\,.$$

Note that

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}\,.$$

Hence, we can compute the partial sums s_n as follows:

$$s_k = \sum_{n=1}^{k} \frac{1}{n(n+1)}$$

$$= \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

$$= \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \dots + \frac{1}{k} - \frac{1}{k+1}$$

$$= 1 - \frac{1}{k+1}\,.$$

Since,

$$\lim_{k \to \infty} s_k = \lim_{k \to \infty} \left(1 - \frac{1}{k+1} \right) = 1\,,$$

the series converges to 1, that is,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1\,.$$

A series of this kind is called a **telescopic sum**, since we can *fold* the entire partial sum together, in such a way that only two terms remain.

Let us go back to the series considered in (8.1).

Example 8.5

Consider the series

$$\sum_{n=1}^{\infty} (-1)^n\,.$$

The partial sums s_k are given by

$$s_k = \sum_{n=1}^{k} (-1)^n = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Therefore s_k diverges, so also the series diverges.

In general, it is a difficult taks to compute the exact value of a series. Therefore, we will mainly focus our effort on determining whether a series converges or not.

The following theorem shows that if the terms in the sequence do not converge to 0, then the series cannot converge.

Theorem 8.6: Necessary Condition for Convergence

Let (a_n) be a sequence in $\mathbb{C}$. If the series

$$\sum_{n=1}^{\infty} a_n$$

converges, then

$$\lim_{n \to \infty} a_n = 0 \,.$$

Proof

Suppose that

$$\sum_{n=1}^{\infty} a_n$$

converges. By definition of convergent series there exists some $s \in \mathbb{C}$ such that

$$\lim_{k \to \infty} s_k = \lim_{k \to \infty} \sum_{n=1}^{k} a_n = s \,.$$

Then also

$$\lim_{k \to \infty} s_{k+1} = s \,.$$

Hence, by the Algebra of Limits in $\mathbb{C}$, we have that

$$\lim_{k \to \infty} (s_{k+1} - s_k) = \lim_{k \to \infty} s_{k+1} - \lim_{k \to \infty} s_k = s - s = 0 \,.$$

Noting that

$$s_{k+1} - s_k = a_{k+1} \,, \quad \forall \, k \in \mathbb{N} \,,$$

we obtain

$$\lim_{k \to \infty} a_k = \lim_{k \to \infty} a_{k+1} = \lim_{k \to \infty} (s_{k+1} - s_k) = 0 \,.$$

Important

Theorem 8.6 is saying that, if
$$\lim_{n\to\infty} a_n \neq 0 \,,$$
then the series
$$\sum_{n=1}^{\infty} a_n$$
does not converge.

Example 8.7

Consider the series
$$\sum_{n=1}^{\infty} (-1)^n \,. \tag{8.2}$$
We have that
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n \neq 0 \,,$$
being (a_n) divergent. Therefore the series at (8.2) diverges by Theorem 8.6.

Example 8.8

Consider the series
$$\sum_{n=1}^{\infty} \frac{n}{5n + 11} \,.$$
Then
$$a_n := \frac{n}{5n + 11} = \frac{1}{5 + \dfrac{11}{n}} \longrightarrow \frac{1}{5} \neq 0 \,.$$
Hence, the series does not converge.

Important

Theorem 8.6 says that if $\sum_{n=1}^{\infty} a_n$ converges, then
$$a_n \to 0 \,.$$
The converse is false: In general the condition $a_n \to 0$ does not guarantee convergence of the associated series, as shown in the example below.

Example 8.9

Consider the series

$$\sum_{n=1}^{\infty} a_n, \quad a_n := \frac{1}{\sqrt{n+1} + \sqrt{n}}.$$

By the Algebra of Limits we have

$$\lim_{n \to \infty} a_n = 0.$$

However the partial sums are given by

$$s_k = \sum_{n=1}^{k} a_n = \sqrt{k+1} - 1.$$

Therefore (s_k) is divergent and so the series is divergent.

Exercise: Prove by induction that

$$\sum_{n=1}^{k} a_n = \sqrt{k+1} - 1.$$

Remark 8.10

It is customary to sum a series starting at $n = 1$. However one could start the sum at any $n = N$ with $N \in \mathbb{N}$. This does not affect the convergence of the series, in the sense that

$$\sum_{n=1}^{\infty} a_n \text{ converges } \iff \sum_{n=N}^{\infty} a_n \text{ converges.}$$

In case of convergence, we would of course have

$$\sum_{n=N}^{\infty} a_n = \sum_{n=1}^{\infty} a_n - (a_1 + ... + a_{N-1}).$$

Example 8.11

We have seen that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

Hence also the series

$$\sum_{n=7}^{\infty} \frac{1}{n(n+1)}$$

converges. However, in this case, the partial sums are given by

$$
\begin{aligned}
s_k &= \sum_{n=7}^{k} \frac{1}{n(n+1)} \\
&= \sum_{n=7}^{k} \left(\frac{1}{n} - \frac{1}{n+1} \right) \\
&= \frac{1}{7} - \frac{1}{8} + \frac{1}{8} - \frac{1}{9} + \dots + \frac{1}{k} - \frac{1}{k+1} \\
&= \frac{1}{7} - \frac{1}{k+1}.
\end{aligned}
$$

Therefore

$$\sum_{n=7}^{\infty} \frac{1}{n(n+1)} = \lim_{k \to \infty} s_k = \frac{1}{7}.$$

8.2 Geometric series

Definition 8.12: Geometric Series in $\mathbb{C}$

Let $x \in \mathbb{C}$. The **geometric series** of ratio x is the series

$$\sum_{n=0}^{\infty} x^n.$$

Geometric series are one of the few types of series that can be explicitly computed, as stated in the following theorem.

Theorem 8.13: Geometric Series Test

Let $x \in \mathbb{C}$. We have:

1. If $|x| < 1$, then the geometric series of ratio x converges, with

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \, . \tag{8.3}$$

2. If $|x| \geq 1$, then the geometric series of ratio x diverges.

Important

Recall that the Geometric Sequence Test does not cover the case $|x| = 1$, since in general the sequence

$$a_n = x^n$$

could be convergent or divergent. However the Geometric Series Test covers the case $|x| = 1$, in which case

$$\sum_{n=0}^{\infty} x^n$$

diverges.

Proof

Part 1. Suppose that $|x| < 1$. By using induction we prove that

$$s_k := \sum_{n=0}^{k} x^n = \frac{1 - x^{k+1}}{1-x} \, , \quad \forall \, k \in \mathbb{N} . \tag{8.4}$$

- Base case: For $k = 0$, we get that
$$s_0 = x^0 = 1 = \frac{1 - x^1}{1 - x} \, .$$

- Induction step: Let $k \in \mathbb{N} \cup \{0\}$ and suppose that
$$s_k = \frac{1 - x^{k+1}}{1 - x} \, .$$

Then,

$$s_{k+1} = s_k + x^{k+1}$$

$$= \frac{1 - x^{k+1}}{1 - x} + x^{k+1}$$

$$= \frac{1 - x^{k+1} + (1 - x)x^{k+1}}{1 - x}$$

$$= \frac{1 - x^{k+1} + x^{k+1} - x^{k+2}}{1 - x}$$

$$= \frac{1 - x^{k+2}}{1 - x},$$

concluding the proof of the inductive step.

By the Principle of Induction formula (8.4) holds for all $k \in \mathbb{N}$. Since $|x| < 1$, by the Geometric Sequence Test we infer

$$\lim_{k \to \infty} x^k = 0.$$

Hence

$$\sum_{n=0}^{\infty} x^n = \lim_{k \to \infty} s_k$$

$$= \lim_{k \to \infty} \frac{1 - x^{k+1}}{1 - x}$$

$$= \lim_{k \to \infty} \frac{1 - x \cdot x^k}{1 - x}$$

$$= \frac{1}{1 - x},$$

where the last equality follows from the Algebra of Limits.

Part 2. Suppose $|x| \geq 1$. Then

$$\lim_{n \to \infty} x^n \neq 0. \tag{8.5}$$

Indeed, suppose by contradiction that $x_n \to 0$. Hence, for $\varepsilon = 1/2$, there exists $N \in \mathbb{N}$ such that

$$|x^n - 0| < \varepsilon = \frac{1}{2}, \quad \forall n \geq \mathbb{N}.$$

However

$$|x^n - 0| = |x^n| = |x|^n \geq 1,$$

which yields

$$1 < \varepsilon = \frac{1}{2},$$

contradiction. Then (8.5) holds and the series

$$\sum_{n=0}^{\infty} x^n$$

diverges by the Necessary Condition in Theorem 8.6.

Let us show some applications of the Geometric Series Test of Theorem 8.13.

Example 8.14

- Since $\left|\frac{1}{2}\right| < 1$ we have

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1 - \frac{1}{2}} = 2$$

- Since $\left|\frac{-3}{2}\right| = \frac{3}{2} > 1$ the series

$$\sum_{n=0}^{\infty} \left(\frac{-3}{2}\right)^n$$

does not converge.

- Since $\left|\frac{-3}{4}\right| = \frac{3}{4} < 1$, we have

$$\sum_{n=0}^{\infty} \left(\frac{-3}{4}\right)^n = \frac{1}{1 - \frac{-3}{4}} = \frac{1}{\frac{7}{4}} = \frac{4}{7}$$

- Since $|-1| = 1$, the series

$$\sum_{n=0}^{\infty} (-1)^n$$

does not converge.

Remark 8.15

If the sum of a Geometric Sries does not start at $n = 0$, we need to tweak the summation formula at (8.3). For example, if $|x| < 1$, and we start the series at $n = 1$, we get

$$\sum_{n=1}^{\infty} x^k = \frac{1}{1 - x} - 1 = \frac{x}{1 - x}.$$

Example 8.16

We have that

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1 \,.$$

The Geometric Series Test of Theorem 8.13 can be applied to complex geometric series as well.

Example 8.17

- Consider the series
$$\sum_{n=0}^{\infty} \frac{1}{(1+i)^n} \,.$$

Since
$$\frac{1}{(1+i)^n} = \left(\frac{1}{1+i}\right)^n$$

and
$$\left|\frac{1}{1+i}\right| = \frac{1}{\sqrt{1^2 + 1^2}} = \frac{1}{\sqrt{2}} < 1 \,,$$

Therefore the series converges by the Geometric Series Test, and

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{1}{(1+i)^n} &= \frac{1}{1 - \frac{1}{1+i}} \\
&= \frac{1}{\frac{1+i-1}{1+i}} \\
&= \frac{1+i}{i} \\
&= \frac{(1+i)i}{i^2} \\
&= \frac{i-1}{-1} \\
&= 1 - i \,.
\end{aligned}$$

- Consider the series
$$\sum_{n=0}^{\infty} \left(\frac{-5-5i}{3+3i}\right)^n \,.$$

Since

$$\left|\frac{-5 - 5i}{3 + 3i}\right| = \frac{|-5 - 5i|}{|3 + 3i|}$$

$$= \frac{\sqrt{(-5)^2 + (-5)^2}}{\sqrt{3^2 + 3^2}}$$

$$= \sqrt{\frac{50}{18}}$$

$$= \frac{5}{3} > 1,$$

the above series does not converge.

- Consider the series

$$\sum_{n=0}^{\infty} \left(\frac{2 + i}{3 - 2i}\right)^n.$$

We have

$$\left|\frac{2 + i}{3 - 2i}\right| = \frac{|2 + i|}{|3 - 2i|}$$

$$= \frac{\sqrt{2^2 + 1^2}}{\sqrt{3^2 + (-2)^2}}$$

$$= \sqrt{\frac{5}{13}} < 1.$$

Therefore the series converges by the Geometric Series Test, and

$$\sum_{n=0}^{\infty} \left(\frac{2 + i}{3 - 2i}\right)^n = \frac{1}{1 - \dfrac{2 + i}{3 - 2i}}$$

$$= \frac{1}{\dfrac{3 - 2i - (2 + i)}{3 - 2i}}$$

$$= \frac{3 - 2i}{1 - 3i}$$

$$= \frac{3 - 2i}{1 - 3i} \frac{1 + 3i}{1 + 3i}$$

$$= \frac{3 - 2i + 9i - 6i^2}{1 - 9i^2}$$

$$= \frac{9 + 7i}{10}$$

$$= \frac{9}{10} + \frac{7}{10}i$$

8.3 Algebra of Limits for Series

We have proven the Algebra of Limit Theorem for sequences in $\mathbb{C}$. A similar results holds for series as well.

Theorem 8.18: Algebra of Limits for Series

Let $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ be sequences in $\mathbb{C}$ and let $c \in \mathbb{C}$. Suppose that

$$\sum_{n=1}^{\infty} a_n = a, \qquad \sum_{n=1}^{\infty} b_n = b.$$

Then:

1. The sum of series is the series of the sums:

$$\sum_{n=1}^{\infty} (a_n \pm b_n) = a \pm b.$$

2. The product of a series with a number obeys

$$\sum_{n=1}^{\infty} c \cdot a_n = c \cdot a.$$

Proof

Part 1. We prove the formula with the $+$ sign, since in the other case the proof is the same. To this end, define the partial sums

$$s_k := \sum_{n=1}^{k} a_n, \quad t_k := \sum_{n=1}^{k} b_n, \quad v_k := \sum_{n=1}^{k} (a_n + b_n).$$

We can write

$$\begin{aligned}
v_k &= \sum_{n=1}^{k} (a_n + b_n) \\
&= (a_1 + b_1) + \ldots + (a_k + b_k) \\
&= (a_1 + \ldots + a_k) + (b_1 + \ldots + b_k) \\
&= s_k + t_k.
\end{aligned}$$

By assumption $s_k \to a$ and $t_k \to b$. Hence, by the Algebra of Limits in $\mathbb{C}$, we infer

$$\sum_{n=1}^{\infty} (a_n + b_n) = \lim_{k \to \infty} v_k$$

$$= \lim_{k \to \infty} (s_k + t_k)$$

$$= \lim_{k \to \infty} s_k + \lim_{k \to \infty} t_k$$

$$= a + b \,.$$

Part 2. Denote the partial sums by

$$s_k := \sum_{n=1}^{k} a_n \,, \quad t_k := \sum_{n=1}^{k} c \cdot a_n \,.$$

We can write

$$t_k = \sum_{n=1}^{k} c \cdot a_k$$

$$= c \cdot a_1 + c \cdot a_2 + \ldots + c \cdot a_k$$

$$= c \cdot (a_1 + a_2 + \ldots + a_k) =$$

$$= c \cdot s_k \,.$$

By assumption $s_k \to a$, so that the Algebra of Limits in $\mathbb{C}$ allows to conclude

$$\sum_{n=1}^{\infty} c \cdot a_n = \lim_{k \to \infty} t_k$$

$$= \lim_{k \to \infty} c \cdot s_k$$

$$= c \cdot a \,.$$

Let us apply the Algebra of Limits for Series to a concrete example.

Example 8.19

Consider the following series

$$\sum_{n=0}^{\infty} \left(2 \left(\frac{1}{3} \right)^n + \left(\frac{2}{3} \right)^n \right) \,.$$

Note that

$$\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2}, \quad \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{1 - \frac{2}{3}} = 3,$$

by the Geometric Series Test of Theorem 8.13. Therefore we can apply the Algebra of Limit for Series to conclude

$$\sum_{n=0}^{\infty} \left(2\left(\frac{1}{3}\right)^n + \left(\frac{2}{3}\right)^n\right) = 2 \cdot \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^k + \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^k$$

$$= 2 \cdot \frac{3}{2} + 3 = 6$$

Important

The Algebra of Limits Theorem 8.18 does not discuss product and quotient of series. The situation becomes more complicated in this case: Indeed, we have

$$(a_1 + a_2) \cdot (a_2 + b_2) = a_1 b_1 + a_2 b_2 + a_1 b_2 + a_2 b_1 \,.$$

Therefore, in general, we can expect

$$\left(\sum_{n=0}^{\infty} a_n\right) \cdot \left(\sum_{n=0}^{\infty} b_n\right) \neq \sum_{n=0}^{\infty} a_n \cdot b_n \,. \tag{8.6}$$

A way to compute

$$\left(\sum_{n=0}^{\infty} a_n\right) \cdot \left(\sum_{n=0}^{\infty} b_n\right)$$

is through the so-called **Cauchy Product** of two series. We do not cover the latter, and the interested reader can refer to Page 82 in [1].
Similarly, we expect that

$$\frac{\displaystyle\sum_{n=0}^{\infty} a_n}{\displaystyle\sum_{n=0}^{\infty} b_n} \neq \sum_{n=0}^{\infty} \frac{a_n}{b_n} \,. \tag{8.7}$$

Let us give two examples to show that formulas (8.6) and (8.7) hold.

Example 8.20

- We know that

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2 \,. \tag{8.8}$$

Therefore

$$\left(\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n\right) \cdot \left(\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n\right) = 2 \cdot 2 = 4 \,.$$

However

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \cdot \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{1-\frac{1}{4}} = \frac{4}{3} \,.$$

Hence

$$\left(\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n\right) \cdot \left(\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n\right) = 4 \neq \frac{4}{3} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \cdot \left(\frac{1}{2}\right)^n \,.$$

- Using (8.8) we have

$$\frac{\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n}{\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n} = \frac{2}{2} = 1 \,.$$

On the other hand

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{\left(\frac{1}{2}\right)^n} = \sum_{n=0}^{\infty} 1 \,,$$

which does not converge by the Necessary Condition, since 1 does not converge to 0. Therefore

$$\frac{\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n}{\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n} \neq \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{\left(\frac{1}{2}\right)^n} \,.$$

8.4 Non-negative series

We now investigate series of which all terms are non-negative. To be precise:

Definition 8.21:   Non-negative series

Let (a_n) be a sequence in $\mathbb{R}$. We call the series

$$\sum_{n=1}^{\infty} a_n$$

a **non-negative series** if

$$a_n \geq 0\,, \quad \forall\, n \in \mathbb{N}\,.$$

The key remark for non-negative series is that the partial sums are increasing.

Lemma 8.22

Let (a_n) be a sequence in $\mathbb{R}$ with

$$a_n \geq 0\,, \quad \forall\, n \in \mathbb{N}\,.$$

Define the partial sums as

$$s_k := \sum_{n=1}^{k} a_n\,.$$

The sequence (s_k) is increasing.

Proof

For all $k \in \mathbb{N}$ we have

$$s_{k+1} = \sum_{n=1}^{k+1} a_n = s_k + a_{k+1} \geq s_k\,,$$

where we used that $a_{k+1} \geq 0$. Therefore (s_k) is increasing.

Therefore, if we have a series with non-negative terms

$$\sum_{n=1}^{\infty} a_n$$

there are only 2 options:

- $\sum_{n=1}^{\infty} a_n$ converges,
- $\sum_{n=1}^{\infty} a_n$ diverges to $+\infty$.

This is because the partial sums (s_k) are increasing. Therefore we have either:

- (s_k) is bounded above: Then s_k converges by the Monotone Convergence Theorem

- (s_k) is not bounded above: Therefore s_k diverges to $+\infty$.

We present several tests for the convergence of non-negative series.

8.4.1 Cauchy Condensation Test

Let us start with the study of the two non-negative series:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \, , \quad \sum_{n=1}^{\infty} \frac{1}{n} \, .$$

Question 8.23

Do the above series converge or diverge?

The answer is that the first series converges, while the second diverges. We prove it in the next two theorems.

Theorem 8.24

The following series converges

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \, .$$

Proof

For $k \in \mathbb{N}$ define the sequence of partial sums

$$s_k := \sum_{n=1}^{k} \frac{1}{n^2} \, .$$

Note that

$$s_k = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots + \frac{1}{k^2}$$

$$= 1 + \frac{1}{2 \cdot 2} + \frac{1}{3 \cdot 3} + \frac{1}{4 \cdot 4} + \dots + \frac{1}{k \cdot k}$$

$$< 1 + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2} + \frac{1}{4 \cdot 3} + \dots + \frac{1}{k \cdot (k-1)}$$

$$= 1 + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{k-1} - \frac{1}{k}\right)$$

$$= 1 + 1 - \frac{1}{k}$$

$$= 2 - \frac{1}{k}$$

$$< 2 \,,$$

showing that s_k is bounded above. Recall that s_k is increasing, by Lemma 8.22. Therefore, by the Monotone Convergence Theorem, we conclude that s_k converges. Hence the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

is convergent.

Theorem 8.25: Harmonic series

The **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent.

Proof

For $k \in \mathbb{N}$ define the sequence of partial sums

$$s_k := \sum_{n=1}^{k} \frac{1}{n} \,.$$

Note that

$$s_2 = 1 + \frac{1}{2}$$

while

$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right)$$
$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right)$$
$$= 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right)$$
$$= 1 + \frac{1}{2} + \frac{1}{2}$$
$$= 1 + 2\left(\frac{1}{2}\right)$$

Similarly

$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$
$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$$
$$= 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right)$$
$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$$
$$= 1 + 3\left(\frac{1}{2}\right)$$

Proceeding in a similar way, for $k \in \mathbb{N}$ we have

$$s_{2^k} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$
$$+ \dots + \left(\frac{1}{2^{k-1} + 1} + \dots + \frac{1}{2^k}\right)$$
$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$$
$$+ \dots + \left(\frac{1}{2^k} + \dots + \frac{1}{2^k}\right)$$
$$= 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) + \dots + 2^{k-1}\left(\frac{1}{2^k}\right)$$
$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2}$$
$$= 1 + k\left(\frac{1}{2}\right)$$

Hence

$$s_{2^k} > 1 + k\left(\frac{1}{2}\right), \quad \forall k \in \mathbb{N},$$

showing that s_{2^k} is unbounded. Therefore s_k is unbounded, and s_k does not converge. Therefore the series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

is divergent.

The proofs of the above theorems inspire the Cauchy Condensation Test.

Theorem 8.26: Cauchy Condensation Test

Let (a_n) be a sequence in $\mathbb{R}$. Suppose that (a_n) is non-negative and decreasing, that is,

$$a_n \geq a_{n+1}, \quad \forall n \in \mathbb{N}.$$

They are equivalent:

1. The series

$$\sum_{n=1}^{\infty} a_n$$

   converges.

2. The series

$$\sum_{n=0}^{\infty} 2^n a_{2^n} = a_1 + 2a_2 + 8a_8 + 16a_{16} + \ldots$$

   converges.

Proof

For $k \in \mathbb{N}$ denote the partial sums by

$$s_k := \sum_{n=1}^{k} a_n, \quad t_k := \sum_{n=0}^{k} 2^n a_{2^n}.$$

Since $a_n \geq 0$, it is immediate to check that the sequences (s_k) and (t_k) are increasing.
Part 1. Assume that the series

$$\sum_{n=0}^{\infty} 2^n a_{2^n}$$

diverges. Hence the sequence (t_k) diverges and therefore (t_k) is not bounded above.

Indeed, suppose (t_k) was bounded above. Since (t_k) is increasing, we would conclude that (t_k) is convergent by the Monotone Convergence Theorem. Contradiction.

We want to estimate s_k from below. To this end, we notice that

$$
\begin{aligned}
s_2 &= a_1 + a_2 \\
&\geq \frac{1}{2} a_1 + a_2 \\
&= \frac{1}{2} (a_1 + 2a_2) \\
&= \frac{1}{2} t_1 \,,
\end{aligned}
$$

where we used that $a_n \geq 0$, and so $a_1 > a_1/2$. Moreover

$$
\begin{aligned}
s_4 &= a_1 + a_2 + (a_3 + a_4) \\
&\geq a_1 + a_2 + (a_4 + a_4) \\
&= a_1 + a_2 + 2a_4 \\
&\geq \frac{1}{2} a_1 + a_2 + 2a_4 \\
&= \frac{1}{2} (a_1 + 2a_2 + 4a_4) \\
&= \frac{1}{2} t_2 \,,
\end{aligned}
$$

where we used also that (a_n) is decreasing. Similar reasoning yields

$$
\begin{aligned}
s_8 &= a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) \\
&\geq a_1 + a_2 + (a_4 + a_4) + (a_8 + a_8 + a_8 + a_8) \\
&= a_1 + a_2 + 2a_4 + 4a_8 \\
&\geq \frac{1}{2} a_1 + a_2 + 2a_4 + 4a_8 \\
&= \frac{1}{2} (a_1 + 2a_2 + 4a_4 + 8a_8) \\
&= \frac{1}{2} t_3 \,.
\end{aligned}
$$

where again we used that (a_n) is decreasing, and $a_n \geq 0$. Iterating, we obtain that for all $k \in \mathbb{N}$ it holds:

$$
\begin{aligned}
s_{2^k} &= a_1 + a_2 + (a_3 + a_4) + (a_5 + a_6 + a_7 + a_8) \\
&\quad + \ldots + (a_{2^{k-1}} + \ldots + a_{2^k}) \\
&\geq a_1 + a_2 + (a_4 + a_4) + (a_8 + a_8 + a_8 + a_8) \\
&\quad + \ldots + (a_{2^k} + \ldots + a_{2^k}) \\
&= a_1 + a_2 + 2a_4 + 4a_8 + \ldots + 2^{k-1}a_{2^k} \\
&\geq \frac{1}{2}a_1 + a_2 + 2a_4 + 4a_8 + \ldots + 2^{k-1}a_{2^k} \\
&= \frac{1}{2}\left(a_1 + 2a_2 + 4a_4 + 8a_8 + \ldots + 2^k a_{2^k}\right) \\
&= \frac{1}{2}t_k .
\end{aligned}
$$

We have shown that

$$
s_{2^k} \geq \frac{1}{2}t_k , \quad \forall k \in \mathbb{N}.
$$

Since (t_k) is not bounded above, we infer that (s_{2^k}) is not bounded above. In particular (s_k) is not bounded, and hence divergent. Thus the series

$$
\sum_{n=1}^{\infty} a_n
$$

diverges.

Part 2. Suppose that the series

$$
\sum_{n=0}^{\infty} 2^n a_{2^n}
$$

converges. Hence the sequence (t_k) converges, and therefore (t_k) is bounded. This means that there exists $M > 0$ such that

$$
|t_k| \leq M , \quad \forall n \in \mathbb{N}.
$$

Since $t_k \geq 0$, the above reads

$$
t_k \leq M , \quad \forall n \in \mathbb{N}.
$$

Fix $k \in \mathbb{N}$ and let $m \in \mathbb{N}$ be such that

$$
k \leq 2^{m+1} - 1 .
$$

In this way

$$
s_k \leq s_{2^{m+1}-1} .
$$

We have

$$
\begin{aligned}
s_{2^{m+1}-1} &= a_1 + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) \\
&\quad + \ldots + (a_{2^m} + \ldots + a_{2^{m+1}-1}) \\
&\leq a_1 + (a_2 + a_2) + (a_4 + a_4 + a_4 + a_4) \\
&\quad + \ldots + (a_{2^m} + \ldots + a_{2^m}) \\
&= a_1 + 2a_2 + 4a_4 + \ldots + 2^m a_{2^m} \\
&= t_m \,,
\end{aligned}
$$

where we used that (a_n) is decreasing. We have then shown

$$
s_k \leq s_{2^{m+1}-1} \leq t_m \leq M \,.
$$

Since M does not depend on k, we conclude that

$$
s_k \leq M \,, \quad \forall\, k \in \mathbb{N} \,.
$$

As $s_k \geq 0$, we conclude that s_k is bounded. Recalling that (s_k) is increasing, by the Monotone Convergence Theorem we infer that (s_k) converges. This proves that the series

$$
\sum_{n=1}^{\infty} a_n
$$

is convergent, ending the proof.

Thanks to the Cauchy Condensation Test of Theorem 8.26 we can prove the following result.

Theorem 8.27: Convergence of p-series

Let $p \in \mathbb{R}$. Consider the p-series

$$
\sum_{n=1}^{\infty} \frac{1}{n^p} \,.
$$

We have:

1. If $p > 1$ the p-series converges.

2. If $p \leq 1$ the p-series diverges.

Proof

The series in question is

$$\sum_{n=1}^{\infty} a_n, \quad a_n := \frac{1}{n^p} \,.$$

Note that (a_n) is decreasing and non-negative. Hence, by the Cauchy Condensation Test of Theorem 8.26, the p-series converges if and only if

$$\sum_{n=0}^{\infty} 2^n a_{2^n}$$

converges. We have

$$\sum_{n=0}^{\infty} 2^n a_{2^n} = \sum_{n=0}^{\infty} 2^{n-np} = \sum_{n=0}^{\infty} (2^{1-p})^n \,,$$

and the latter is a Geometric Series of ratio

$$x := 2^{1-p} \,.$$

By the Geometric Series Test, we have convergence of

$$\sum_{n=0}^{\infty} (2^{1-p})^n$$

if and only if

$$|x| < 1 \,.$$

The above is equivalent to

$$2^{1-p} < 1 = 2^0 \quad \Longleftrightarrow \quad 1 - p < 0 \quad \Longleftrightarrow \quad p > 1 \,.$$

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if and only if $p > 1$, ending the proof.

The following is another notable application of the Cauchy Condensation Test.

Theorem 8.28

Let $p \in \mathbb{R}$. Consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n\,(\log n)^p} \,.$$

We have:

1. If $p > 1$ the series converges.

2. If $p \leq 1$ the series diverges.

Proof

The series in question is

$$\sum_{n=2}^{\infty} a_n, \quad a_n := \frac{1}{n \left(\log n\right)^p}.$$

Note that (a_n) is non-negative and decreasing. Therefore we can apply the Cauchy Condensation Test to conclude that the above series is convergent if and only if the series

$$\sum_{n=1}^{\infty} 2^n a_{2^n}$$

is convergent. We have

$$2^n a_{2^n} = 2^n \frac{1}{2^n \left(\log 2^n\right)^p} = \frac{1}{n^p \, \log 2}$$

so that

$$\sum_{n=1}^{\infty} 2^n a_{2^n} = \frac{1}{\log 2} \sum_{n=1}^{\infty} \frac{1}{n^p}.$$

The latter is a p-series, which by Theorem 8.27 converges if and only if $p > 1$. Hence

$$\sum_{n=2}^{\infty} \frac{1}{n \left(\log n\right)^p}$$

converges if and only if $p > 1$, and the proof is concluded.

8.4.2 Comparison Test

Another really useful result to study the convergence of non-negative series is the Comparison Test.

Theorem 8.29: Comparison test

Let $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ be non-negative sequences. Suppose that there exists $N \in \mathbb{N}$ such that

$$a_n \leq b_n, \quad \forall n \geq N.$$

They hold:

1. If $\sum_{n=1}^{\infty} b_n$ converges, then also $\sum_{n=1}^{\infty} a_n$ converges.

2. If $\sum_{n=1}^{\infty} a_n$ diverges, then also $\sum_{n=1}^{\infty} b_n$ diverges.

Proof

Part 1. Define the partial sums starting at $n = N$

$$s_k := \sum_{n=N}^{k} a_k, \quad t_k := \sum_{n=N}^{k} b_k.$$

Suppose that

$$\sum_{n=1}^{\infty} b_n$$

converges. Hence also the series

$$\sum_{n=N}^{\infty} b_n$$

converges. Then (t_k) is a convergent sequence, which implies that (t_k) is bounded, and hence bounded above. We have that s_k is bounded above: Indeed, using the assumption, we have

$$\begin{aligned} s_k &= a_N + a_{N+1} + \ldots + a_k \\ &\leq b_N + b_{N+1} + \ldots + b_k \\ &= t_k, \end{aligned}$$

which reads

$$s_k \leq t_k, \quad \forall\, k \geq N.$$

Therefore (s_k) is bounded above, being (t_k) bounded above. Recall that s_k is increasing, by Lemma 8.22. By the Monotone Convergence Theorem we conclude that s_k is convergent, showing that the series

$$\sum_{n=N}^{\infty} a_n$$

converges. Hence also the series

$$\sum_{n=1}^{\infty} a_n$$

converges, concluding the proof of Point 1.

Part 2. Note that Point 2 is the contrapositive of Point 1, and hence it holds.

Let us give two applications of the Comparison Test.

Example 8.30

- Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 3n - 1} \,. \tag{8.9}$$

Since $3n - 1 \geq 0$ for all $n \in \mathbb{N}$, we get

$$\frac{1}{n^2 + 3n - 1} \leq \frac{1}{n^2} \,, \quad \forall \, n \in \mathbb{N} \,.$$

By Theorem 8.27 the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges. Therefore also the series at (8.9) converges by the Comparison Test in Theorem 8.28.

- Consider the series

$$\sum_{n=0}^{\infty} \frac{3^n + 6n + \dfrac{1}{n+1}}{2^n} \,. \tag{8.10}$$

Note that

$$\frac{3^n + 6n + \dfrac{1}{n+1}}{2^n} \geq \frac{3^n}{2^n} = \left(\frac{3}{2}\right)^n \,, \quad \forall \, n \in \mathbb{N} \,.$$

Since $\left|\frac{3}{2}\right| = \frac{3}{2} > 1$, the series

$$\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n$$

diverges by the Geometric Series Test in Theorem 8.13. Therefore, by the Comparison Test, also the series at (8.10) diverges.

8.4.3 Limit Comparison Test

To apply the Comparison Test to the series

$$\sum_{n=1}^{\infty} a_n \,,$$

one needs to find another sequence (b_n) such that

$$a_n \leq b_n \,, \quad \forall \, n \geq N \,,$$

or

$$b_n \leq a_n \,, \quad \forall \, n \geq N \,.$$

This is not always possible. However, one might be able to show that

$$L = \lim_{n \to \infty} \frac{a_n}{b_n}$$

for some $L \in \mathbb{R}$. In this case, the series of (a_n) and (b_n) can still be compared, in the sense specified in the below theorem.

Theorem 8.31: Limit Comparison Test

Let (a_n) and (b_n) be sequences such that

$$a_n \geq 0, \quad b_n > 0, \quad \forall n \in \mathbb{N}.$$

Suppose there exists $L \in \mathbb{R}$ such that

$$L = \lim_{n \to \infty} \frac{a_n}{b_n}.$$

They hold:

1. If $0 < L < \infty$, then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=1}^{\infty} b_n \text{ converges.}$$

2. If $L = 0$, then

   - If $\sum_{n=1}^{\infty} b_n$ converges also $\sum_{n=1}^{\infty} a_n$ converges,
   - If $\sum_{n=1}^{\infty} a_n$ diverges also $\sum_{n=1}^{\infty} b_n$ diverges.

Proof

Part 1. Suppose that $0 < L < 1$. Set

$$\varepsilon := \frac{L}{2}.$$

Since $\varepsilon > 0$ and $a_n / b_n \to L$, there exists $N \in \mathbb{N}$ such that

$$\left| \frac{a_n}{b_n} - L \right| < \varepsilon, \quad \forall n \geq N.$$

The above is equivalent to

$$L - \varepsilon < \frac{a_n}{b_n} < \varepsilon + L, \quad \forall n \geq N.$$

Since $\varepsilon = L/2$, we get

$$\frac{L}{2} < \frac{a_n}{b_n} < \frac{3L}{2}, \quad \forall n \geq N,$$

or equivalently,

$$\frac{L}{2} b_n < a_n < \frac{3L}{2} b_n, \quad \forall n \geq N.$$

We are now ready to prove the main claim:

- Suppose that

$$\sum_{n=1}^{\infty} a_n$$

  converges. Then also

$$\sum_{n=N}^{\infty} a_n \, ,$$

  since we are only discarding a finite number of terms. As

$$\frac{L}{2} b_n \leq a_n \, , \quad \forall \, n \geq N \, ,$$

  it follows from the Comparison Test in Theorem 8.28 that the series

$$\sum_{n=N}^{\infty} \frac{L}{2} b_n \, .$$

  converges. Since $L/2$ is a constant, we also conclude that

$$\sum_{n=1}^{\infty} b_n$$

  converges.

- Suppose that

$$\sum_{n=1}^{\infty} b_n$$

  converges. Then also

$$\sum_{n=N}^{\infty} \frac{3L}{2} b_n$$

  converges. Since

$$a_n < \frac{3L}{2} b_n \, , \quad \forall \, n \geq N \, ,$$

  by the Comparison Test we infer that

$$\sum_{n=N}^{\infty} a_n$$

  converges. Therefore, also

$$\sum_{n=1}^{\infty} a_n$$

  converges.

Part 2. Suppose that $L = 0$. Note that the second bullet point is the contrapositive of the first. Hence we only need to show the first bullet point. Let $\varepsilon = 1$. Since $a_n/b_n \to 0$, there exists $N \in \mathbb{N}$ such that

$$\left| \frac{a_n}{b_n} - 0 \right| < \varepsilon = 1 \,, \quad \forall\, n \geq N \,.$$

Therefore

$$a_n < b_n \,, \quad \forall\, n \geq N \,.$$

The thesis follows immediately by the Comparison Test in Theorem 8.28.

Important

It might happen that

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \infty \,.$$

In this case it can be helpful to interchange the roles of a_n and b_n, since then

$$\lim_{n \to \infty} \frac{b_n}{a_n} = 0 \,.$$

Let us give a few applications of the Limit Comparison Tets.

Example 8.32

The following series converges

$$\sum_{n=1}^{\infty} \frac{2n^3 + 5n + 1}{7n^6 + 2n + 5} \,.$$

Proof. Set

$$a_n := \frac{2n^3 + 5n + 1}{7n^6 + 2n + 5} \,, \quad b_n := \frac{1}{n^3} \,.$$

We have

$$L := \lim_{n \to \infty} \frac{a_n}{b_n}$$

$$= \lim_{n \to \infty} \frac{2n^3 + 5n + 1}{7n^6 + 2n + 5} \bigg/ \frac{1}{n^3}$$

$$= \lim_{n \to \infty} \frac{2n^6 + 5n^4 + n^3}{7n^6 + 2n + 5}$$

$$= \lim_{n \to \infty} \frac{2 + \dfrac{5}{n^2} + \dfrac{1}{n^3}}{7 + \dfrac{2}{n^5} + \dfrac{5}{n^6}} = \frac{2}{7}\,.$$

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

converges, being a p-series with $p = 3 > 1$. Since $L = \frac{2}{7} > 0$, also the series of interest converges, by the Limit Comparison Test.

Example 8.33

Prove that the following series diverges

$$\sum_{n=1}^{\infty} \frac{n + \cos(n)}{n^2}\,.$$

Proof. We expect the terms in the series to behave like $1/n$ for large n. Hence we set

$$a_n := \frac{n + \cos(n)}{n^2}\,, \quad b_n = \frac{1}{n}\,.$$

We compute

$$L := \frac{a_n}{b_n}$$

$$= \lim_{n \to \infty} \frac{n + \cos(n)}{n^2} \bigg/ \frac{1}{n}$$

$$= \lim_{n \to \infty} \frac{n^2 + n\cos(n)}{n^2}$$

$$= \lim_{n \to \infty} \left(1 + \frac{\cos(n)}{n}\right)$$

Since

$$-1 \le \cos(n) \le 1\,,$$

we obtain
$$-\frac{1}{n} \leq \frac{\cos(n)}{n} \leq \frac{1}{n}\,.$$

As both $-\frac{1}{n} \to 0$ and $\frac{1}{n} \to 0$, by the Squeeze Theorem
$$\frac{\cos(n)}{n} \longrightarrow 0\,.$$

Hence
$$L = \lim_{n\to\infty} \left(1 + \frac{\cos(n)}{n}\right) = 1\,.$$

The harmonic series
$$\sum_{n=1}^{\infty} \frac{1}{n}$$

does not converge. Since $L = 1 > 0$, the series of interest diverges by the Limit Comparison Test.

Example 8.34

Prove that the series
$$\sum_{n=1}^{\infty} \left(1 - \cos\left(\frac{1}{n}\right)\right)$$

converges.

Solution. Since
$$\cos\left(\frac{1}{n}\right) \leq 1\,,$$

the above is a non-negative series. Recall the limit
$$\lim_{n\to\infty} \frac{1 - \cos(a_n)}{(a_n)^2} = \frac{1}{2}\,,$$

where (a_n) is a sequence in $\mathbb{R}$ such that $a_n \to 0$ and
$$a_n \neq 0 \quad \forall\, n \in \mathbb{N}\,.$$

In particular, for $a_n = 1/n$, we obtain
$$\lim_{n\to\infty} n^2 \left(1 - \cos\left(\frac{1}{n}\right)\right) = \frac{1}{2}\,.$$

Set
$$b_n := 1 - \cos\left(\frac{1}{n}\right)\,, \quad c_n := \frac{1}{n^2}\,.$$

We have

$$L := \lim_{n \to \infty} \frac{b_n}{c_n}$$

$$= \lim_{n \to \infty} n^2 \left(1 - \cos\left(\frac{1}{n}\right)\right)$$

$$= \frac{1}{2}.$$

Note that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges, being a p-series with $p > 2$. Therefore, since $L = 1/2 > 0$, also the series

$$\sum_{n=1}^{\infty} \left(1 - \cos\left(\frac{1}{n}\right)\right)$$

converges, by the Limit Comparison Test.

Sometimes the Limit Comparison Test fails, but the Comparison Test works.

Example 8.35

Consider the series

$$\sum_{n=1}^{\infty} \frac{1 + \sin(n)}{n^2}.$$

Since

$$\sin(n) \geq -1,$$

the above is a non-negative series. We expect this series to behave similar to

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

However

$$\frac{1 + \sin(n)}{n^2} \bigg/ \frac{1}{n^2} = 1 + \sin(n)$$

does not converge. Hence, we cannot use the Limit Comparison Test. In alternative, we note that

$$\frac{1 + \sin(n)}{n^2} \leq \frac{2}{n^2}, \quad \forall\, n \in \mathbb{N}.$$

The series

$$\sum_{n=1}^{\infty} \frac{2}{n^2}$$

converges, being a p-series with $p = 2 > 1$. Therefore also

$$\sum_{n=1}^{\infty} \frac{1 + \sin(n)}{n^2}$$

converges, by the Comparison Test of Theorem 8.28.

8.4.4 Ratio Test for positive series

The Ratio Test can be generalized to series. Notice that in this case the terms of the series need to be positive.

Theorem 8.36: Ratio Test for positive series

Let (a_n) be a sequence in $\mathbb{R}$ such that

$$a_n > 0, \quad \forall\, n \in \mathbb{N}.$$

1. Suppose that the following limit exists:

$$L := \lim_{n \to \infty} \frac{a_{n+1}}{a_n}.$$

   They hold:
   - If $L < 1$ then $\sum_{n=1}^{\infty} a_n$ converges.
   - If $L > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.

2. Suppose that there exists $N \in \mathbb{N}$ and $L > 1$ such that

$$\frac{a_{n+1}}{a_n} \geq L, \quad \forall\, n \geq N.$$

   Then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Proof

Part 1. Let

$$L := \lim_{n \to \infty} \frac{a_{n+1}}{a_n}.$$

- Suppose that $L < 1$. Therefore there exists r such that

$$L < r < 1\,.$$

Define

$$\varepsilon := r - L\,,$$

so that $\varepsilon > 0$. By the convergence $a_{n+1}/a_n \to L$ there exists $N \in \mathbb{N}$ such that

$$\left| \frac{a_{n+1}}{a_n} - L \right| < \varepsilon = r - L\,, \quad \forall\, n \geq N\,.$$

In particular

$$\frac{a_{n+1}}{a_n} - L < r - L\,, \quad \forall\, n \geq N\,,$$

which implies

$$a_{n+1} < r\, a_n\,, \quad \forall\, n \geq N\,.$$

Applying $n - N$ times the above estimate we get

$$0 < a_n < r\, a_{n-1} < \dots < r^{n-N}\, a_N\,, \quad \forall\, n \geq N\,. \tag{8.11}$$

Note that the series of $r^{n-N}\, a_N$ converges, since

$$\sum_{n=N}^{\infty} r^{n-N}\, a_N = a_N \sum_{k=0}^{\infty} r^k = a_N \frac{1}{1-r}\,,$$

where the last equality follows because $\sum_{k=0}^{\infty} r^k$ is a geometric series and $0 < r < 1$. Since (8.11) holds, by the Comparison Test in Theorem 8.28 we conclude that the series

$$\sum_{n=N}^{\infty} a_n$$

converges. Therefore also the series

$$\sum_{n=1}^{\infty} a_n$$

converges, ending the proof in the case $L < 1$.

- Suppose $L > 1$: Then, by the Ratio Test for sequences, it follows that a_n diverges. Therefore

$$\lim_{n \to \infty} a_n \neq 0\,,$$

and the series $\sum_{n=1}^{\infty}$ diverges by the Necessary Condition, see Theorem 8.6.

Part 2. Suppose there exists $L > 1$ and $N \in \mathbb{N}$ such that

$$\frac{a_{n+1}}{a_n} \geq L\,, \quad \forall\, n \geq N\,.$$

By the Ratio Test for sequences it follows that a_n diverges. Therefore $\sum_{n=1}^{\infty}$ diverges by the Necessary Condition.

Example 8.37

Consider the series

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$$

Let

$$a_n = \frac{(n!)^2}{(2n)!}$$

and compute

$$
\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{((n+1)!)^2}{(2(n+1))!} \Big/ \frac{(n!)^2}{(2n)!} \\
&= \frac{((n+1)!)^2}{(n!)^2} \frac{(2n)!}{(2(n+1))!} \\
&= \left(\frac{(n+1)!}{n!} \right)^2 \frac{(2n)!}{(2n+2)!} \\
&= \frac{(n+1)^2}{(2n+2)(2n+1)} \\
&= \frac{\left(1 + \dfrac{1}{n} \right)^2}{\left(2 + \dfrac{2}{n} \right) \left(2 + \dfrac{1}{n} \right)}
\end{aligned}
$$

Therefore

$$L = \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{\left(1 + \dfrac{1}{n} \right)^2}{\left(2 + \dfrac{2}{n} \right) \left(2 + \dfrac{1}{n} \right)} = \frac{1}{4}.$$

Since $L = 1/4 < 1$, be the Ratio Test for Series we conclude that the series of interest converges.

Important

Like with the Ratio Test for sequences, the case $L = 1$ is not covered by Theorem 8.36. This is because, in this case, the series

$$\sum_{n=1}^{\infty} a_n$$

might be convergent or divergent, as shown in the next example.

Example 8.38

- Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

Setting $a_n = 1/n$, we have

$$L = \lim_{n \to \infty} \frac{a_{n+1}}{a_n}$$

$$= \lim_{n \to \infty} \frac{1}{n+1} \bigg/ \frac{1}{n}$$

$$= \lim_{n \to \infty} \frac{n}{n+1} = 1.$$

Therefore $L = 1$ and we cannot apply the Ratio Test. However the series in question diverges, being the harmonic series.

- Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Setting $a_n = 1/n^2$, we have

$$L = \lim_{n \to \infty} \frac{a_{n+1}}{a_n}$$

$$= \lim_{n \to \infty} \frac{1}{(n+1)^2} \bigg/ \frac{1}{n^2}$$

$$= \lim_{n \to \infty} \frac{n^2}{n^2 + 2n + 1} = 1.$$

Therefore $L = 1$ and we cannot apply the Ratio Test. However the series in question diverges, being a p-series with $p = 2 > 1$.

The Ratio Test can often be combined with other convergence tests, as seen in the following example.

Example 8.39

Using the Cauchy Condensation Test and the Ratio Test, prove that the series below converges

$$\sum_{n=1}^{\infty} \frac{\log(n)}{n^2}.$$

Solution. With $a_n = \log n / n^2$, we have

$$\sum_{n=0}^{\infty} 2^n a_{2^n} = \sum_{n=2}^{\infty} 2^n \frac{\log(2^n)}{(2^n)^2} = \log(2) \sum_{n=2}^{\infty} \frac{n}{2^n}$$

This converges according to the Ratio Test: With $b_n = n/2^n$, we have

$$\frac{b_{n+1}}{b_n} = \frac{n+1}{2^{n+1}} \bigg/ \frac{n}{2^n} = \frac{n+1}{2n} \longrightarrow \frac{1}{2} < 1 \,.$$

Hence $\sum_{n=1}^{\infty} \frac{\log n}{n^2}$ converges by the Cauchy Condensation Test.

8.5 General series

In the previous section we presented several tests for non-negative series. For non-negative series we shows that the partial sums (s_k) are increasing, see Lemma 8.22. This makes non-negative terms series relatively easy to study.

When a series contains both positive and negative terms, the partial sums (s_k) might oscillate, making the series harder to study. In this section we present some tests for general series in $\mathbb{C}$.

8.5.1 Absolute Convergence Test

To study general series, we introduce a stronger notions of convergence, known as **absolute convergence**.

Definition 8.40:   Absolute convergence

Let (a_n) be a sequence in $\mathbb{C}$. The series $\sum_{n=1}^{\infty} a_n$ is said to **converge absolutely** if the non-negative series

$$\sum_{n=1}^{\infty} |a_n|$$

converges.

Let us show that absolute convergence implies convergence.

Theorem 8.41: Absolute Convergence Test

Let (a_n) be a sequence in $\mathbb{C}$. If the series $\sum_{n=1}^{\infty} a_n$ converge absolutely, then the series converges.

Before proceeding with the proof, we introduce some notation.

Notation 8.42: Positive and negative part

For a number $x \in \mathbb{R}$, we define

$$x^+ := \begin{cases} x & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad x^- := \begin{cases} -x & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}$$

These are called the **positive part** and **negative part** of x, respectively. Note that

$$x^+ \geq 0, \quad x^- \geq 0.$$

Moreover they hold

$$x = x^+ - x^-, \quad |x| = x^+ + x^-.$$

The above relations are easy to check, and the proof is omitted. In particular, it holds that

$$x^+ \leq |x|, \quad x^- \leq |x|.$$

Proof: Proof of Theorem 8.41

Part 1. Suppose first that (a_n) is a sequence in $\mathbb{R}$ such that

$$\sum_{n=1}^{\infty} |a_n|$$

converges. Since

$$0 \leq a_n^+ \leq |a_n|, \quad \forall n \in \mathbb{N},$$

we can use the Comparison Test for non-negative series (Theorem 8.28) and conclude that the series

$$\sum_{n=1}^{\infty} a_n^+.$$

converges. Similarly, we have

$$0 \leq a_n^- \leq |a_n|, \quad \forall n \in \mathbb{N}.$$

Therefore also the series

$$\sum_{n=1}^{\infty} a_n^-$$

converges by the Comparison Test. Since

$$a_n = a_n^+ - a_n^-, \quad \forall\, n \in \mathbb{N},$$

we can use the Algebra of Limits for series (Theorem 8.18) to conclude that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(a_n^+ - a_n^-\right) = \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n^-$$

converges.

Part 2. Suppose now that (a_n) is a sequence in $\mathbb{C}$ such that

$$\sum_{n=1}^{\infty} |a_n|$$

converges. Let $x_n, y_n \in \mathbb{R}$ denote the real and imaginary part of a_n. Therefore

$$|x_n| = \sqrt{x_n^2} \leq \sqrt{x_n^2 + y_n^2} = |a_n|, \quad \forall\, n \in \mathbb{N}.$$

Therefore the series

$$\sum_{n=1}^{\infty} |x_n|$$

converges by the Comparison Test for non-negative series (Theorem 8.28). Since (x_n) is a real sequence, from Part 1 of the proof we have that the series

$$\sum_{n=1}^{\infty} x_n$$

converges. Arguing in the same way for the imaginary part y_n we conclude that also

$$\sum_{n=1}^{\infty} y_n$$

converges. Finally, by the Algebra of Limits in $\mathbb{C}$, we get

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (x_n + iy_n) = \sum_{n=1}^{\infty} x_n + i \sum_{n=1}^{\infty} y_n,$$

proving that $\sum_{n=1}^{\infty} a_n$ converges.

Example 8.43

The series

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

does not converge absolutely, since

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{1}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

doesn't converge, being the harmonic series.

Example 8.44

Prove that the following series converges

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2 - 5n + 2}{n^4} \, .$$

Solution. We have

$$\left| (-1)^n \frac{n^2 - 5n + 2}{n^4} \right| = \frac{\left| n^2 - 5n + 2 \right|}{n^4} \leq \frac{n^2 + 5n + 2}{n^4}$$

where we used the triangle inequality. Note that

$$\frac{n^2 + 5n + 2}{n^4} \bigg/ \frac{1}{n^2} = \frac{n^4 + 5n^3 + 2n^2}{n^4}$$

$$= 1 + \frac{5}{n} + \frac{2}{n^2} \longrightarrow 1$$

The series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges, being a *p*-series with $p = 2$. Hence, also

$$\sum_{n=1}^{\infty} \frac{n^2 + 5n + 2}{n^4}$$

converges by the Limit Comparison Test for non-negative series (Theorem 8.31). Since

$$0 \leq \left| (-1)^n \frac{n^2 - 5n + 2}{n^4} \right| = \frac{\left| n^2 - 5n + 2 \right|}{n^4}$$

$$\leq \frac{n^2 + 5n + 2}{n^4} \, ,$$

the series

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{n^2 - 5n + 2}{n^4} \right|$$

converges by the Comparison Test for non-negative series (Theorem 8.28). This shows the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2 - 5n + 2}{n^4}$$

converges by the Absolute Convergence Test.

8.5.2 Ratio Test for general series

As an application of the Absolute Convergence Test we obtain the Ratio Test for general series.

Theorem 8.45: Ratio Test for general series

Let (a_n) be a sequence in $\mathbb{C}$, such that

$$a_n \neq 0 \quad \forall n \in \mathbb{N}.$$

1. Suppose that the following limit exists:

$$L := \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

   They hold:

   - If $L < 1$ then $\sum_{n=1}^{\infty} a_n$ converges absolutely, and hence converges.
   - If $L > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.

2. Suppose that there exists $N \in \mathbb{N}$ and $L > 1$ such that

$$\left| \frac{a_{n+1}}{a_n} \right| \geq L, \quad \forall n \geq N.$$

   Then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Proof

Part 1. Let

$$b_n := |a_n|,$$

so that

$$L := \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{b_{n+1}}{b_n} \,.$$

- Suppose that $L < 1$. Since (b_n) is a sequence with non-negative terms, we have that

$$\sum_{n=1}^{\infty} b_n$$

converges by the Ratio Test for non-negative series, see Theorem 8.36. Since, by definition

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} |a_n|$$

also the latter series converges, i.e., $\sum_{n=1}^{\infty} a_n$ converges absolutely. In particular $\sum_{n=1}^{\infty} a_n$ converges, by the Absolute Convergence Test in Theorem 8.41.

- Suppose that $L > 1$. Then the sequence (a_n) diverges by the Ratio Test for sequences. Hence the series

$$\sum_{n=1}^{\infty} a_n$$

diverges by the Necessary Condition in Theorem 8.6.

Part 2. If there exists $N \in \mathbb{N}$ and $L > 1$ such that

$$\left| \frac{a_{n+1}}{a_n} \right| \geq L, \quad \forall\, n \geq N,$$

then the sequence (a_n) diverges by the Ratio Test for sequences, and we conclude as above.

Example 8.46

Prove that the below series converges

$$\sum_{n=1}^{\infty} \frac{(4 - 3i)^n}{(n + 1)!}$$

Solution. Set

$$a_n := \frac{(4 - 3i)^n}{(n + 1)!} \,.$$

Then

$$L := \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \left| \frac{(4-3i)^{n+1}}{((n+1)+1)!} \bigg/ \frac{(4-3i)^n}{(n+1)!} \right|$$

$$= \frac{|4-3i|^{n+1}}{|4-3i|^n} \frac{(n+1)!}{(n+2)!}$$

$$= \frac{5}{n+2}$$

$$= 0$$

By the Ratio Test we conclude that the series $\sum_{n=1}^{\infty} a_n$ converges absolutely, and hence converges.

8.5.3 Exponential function and Euler's Number

We have already encountered the exponential function in the Euler's identity

$$e^{i\theta} = \cos(\theta) + i\sin(\theta), \quad \theta \in \mathbb{R}.$$

Using the Ratio Test for general series, we can give a precise definition for e^z with $z \in \mathbb{C}$. We start by studying convergence of the exponential series.

Theorem 8.47: Exponential series

Let $z \in \mathbb{C}$. The **exponential series**

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

converges absolutely.

Proof

Set

$$a_n = \frac{z^n}{n!}.$$

Then

$$L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

$$= \lim_{n \to \infty} \left| \frac{z^{n+1}}{(n+1)!} \middle/ \frac{z^n}{n!} \right|$$

$$= \lim_{n \to \infty} \frac{|z|^{n+1}}{|z|^n} \frac{n!}{(n+1)!}$$

$$= \lim_{n \to \infty} \frac{|z|}{n+1}$$

$$= |z| \cdot \lim_{n \to \infty} \frac{1}{n+1} = 0 \,.$$

Therefore the series converges absolutely by the Ratio Test in Theorem 8.45.

Definition 8.48: Exponential function

Define the exponential function

$$\exp : \; \mathbb{C} \to \mathbb{C}, \quad \exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

for $z \in \mathbb{C}$. We denote

$$e^z := \exp(z), \quad e := e^1 \,.$$

Remark 8.49

1. Using the definition of e^z, one can show the usual properties of exponentials, such that

$$e^{x+y} = e^x e^y, \quad \frac{d}{dx} e^x = e^x \,.$$

2. We had defined

$$e := \lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n \,.$$

Using the binomial theorem one can prove that

$$\lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n = \sum_{n=0}^{\infty} \frac{1}{n!} \,.$$

8.5.4 Conditional convergence

Some series do not converge absolutely, but still converge. Such series are said to converge conditionally.

Definition 8.50: Conditional convergence

Let (a_n) be a sequence in $\mathbb{C}$. We say that the series

$$\sum_{n=1}^{\infty} a_n$$

converges **conditionally** if it converges, but it does not converge absolutely.

In practice conditional convergence means that the convergence of the series depends on the order in which we perform the summation. Changing the order of summation of a series is called rearrangement.

Definition 8.51: Rearrangement of a series

Let (a_n) be a sequence in $\mathbb{C}$. We define:

  - A **permutation** is a bijection $\sigma : \mathbb{N} \to \mathbb{N}$.
  - A **rearrangement** of the series $\sum_{n=1}^{\infty} a_n$ is a series

$$\sum_{n=1}^{\infty} a_{\sigma(n)}$$

  for some permutation σ.

If a series of complex numebers converges absolutely, then all its rearrangements converge to the same limit.

Theorem 8.52

Let (a_n) be a sequence in $\mathbb{C}$ such that

$$\sum_{n=1}^{\infty} |a_n|$$

converges. For any permutation σ we have

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = \sum_{n=1}^{\infty} a_n .$$

For a proof, see Theorem 3.55 in [3]. A very surprising result is the following: If a series of real numbers converges conditionally, then the series can be rearranged to converge to any real number.

Theorem 8.53: Riemann rearrangement Theorem

Let (a_n) be a real series such that

$$\sum_{n=1}^{\infty} a_n$$

converges conditionally. Let

$$L \in \mathbb{R} \quad \text{or} \quad L = \pm\infty.$$

There exists a permutation σ such that the corresponding rearrangement $\sum_{n=1}^{\infty} a_{\sigma(n)}$ converges conditionally to L, that is,

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = L.$$

For a proof, we refer the reader to Theorem 3.54 in [3].

8.5.5 Dirichlet and Alternating Series Tests

There are very few conditional convergence tests available. We present the Dirichlet Test and the Alternating Series Test.

Theorem 8.54: Dirichlet Test

Let (c_n) be a sequence in $\mathbb{C}$ and (q_n) a sequence in $\mathbb{R}$. Suppose that

- q_n is decreasing,
- $q_n \to 0$,
- $q_n \geq 0$ for all $n \in \mathbb{N}$.
- Suppose there exists $M > 0$ such that

$$\left| \sum_{n=1}^{k} c_n \right| \leq M, \quad \forall\, k \in \mathbb{N}.$$

Then the following series converges

$$\sum_{n=1}^{\infty} c_n q_n$$

Proof

Define the partial sums

$$s_k := \sum_{n=1}^{k} c_n, \quad \forall\, k \in \mathbb{N}.$$

By assumption it holds

$$|s_k| \leq M, \quad \forall\, k \in \mathbb{N}. \tag{8.12}$$

Note that

$$c_1 = s_1$$
$$c_2 = s_2 - s_1$$
$$\dots\dots$$
$$c_n = s_n - s_{n-1}.$$

Therefore

$$\sum_{n=1}^{k} c_k q_k = c_1 q_1 + c_2 q_2 + \dots + c_k q_k$$

$$= s_1 q_1 + (s_2 - s_1) q_2 + \dots + (s_k - s_{k-1}) q_k$$
$$= s_1 (q_1 - q_2) + s_2 (q_2 - q_3) + \dots + s_{k-1}(q_{k-1} - q_k) + s_k q_k$$
$$= \left(\sum_{n=1}^{k-1} s_n (q_n - q_{n+1}) \right) + s_k q_k$$

Since (s_k) is bounded and $q_k \to 0$, we conclude that

$$s_k q_k \to 0.$$

Further, notice that

$$q_n - q_{n+1} \geq 0,$$

since q_n is decreasing. Therefore, using (8.12), we get

$$|s_n||q_n - q_{n+1}| = |s_n|(q_n - q_{n+1})$$
$$\leq M(q_n - q_{n+1})$$

for all $n \in \mathbb{N}$. Note that

$$\sum_{n=1}^{k-1} M(q_n - q_{n+1}) = M(q_1 - q_k).$$

Since $q_k \to 0$ as $k \to \infty$, we conclude that

$$\sum_{n=1}^{\infty} M(q_n - q_{n+1}) = M q_1.$$

Hence, by the Comparison Test for non-negative series, we infer that

$$\sum_{n=1}^{\infty} |s_n||q_n - q_{n+1}|.$$

In particular the series

$$\sum_{n=1}^{\infty} s_n(q_n - q_{n+1})$$

converges by the Absolute Convergence Test. Since we have shown

$$\sum_{n=1}^{k} c_k q_k = \left(\sum_{n=1}^{k-1} s_n(q_n - q_{n+1}) \right) + s_k q_k$$

and $s_k q_k \to 0$, we conclude that

$$\sum_{n=1}^{\infty} c_k q_k$$

converges.

Example 8.55

Let $\theta \in \mathbb{R}$, with

$$\theta \neq 2k\pi, \quad \forall\, k \in \mathbb{Z}.$$

Prove that the below series are conditionally convergent

$$\sum_{n=1}^{\infty} \frac{e^{i\theta n}}{n}, \quad \sum_{n=1}^{\infty} \frac{\cos(\theta n)}{n}, \quad \sum_{n=1}^{\infty} \frac{\sin(\theta n)}{n}.$$

Solution. Clearly the series

$$\sum_{n=1}^{\infty} \frac{e^{i\theta n}}{n}$$

does not converge absolutely, since

$$\sum_{n=1}^{\infty} \left| \frac{e^{i\theta n}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges, being the Harmonic Series. Let us prove convergence of $\sum_{n=1}^{\infty} \frac{e^{i\theta n}}{n}$. Define the sequences

$$c_n := e^{i\theta n}, \quad q_n := \frac{1}{n}.$$

We have that q_n is decreasing, $q_n \to 0$ and $q_n \geq 0$. Let us prove that there exists $M > 0$ such that

$$\left| \sum_{n=1}^{k} e^{i\theta n} \right| \leq M, \quad \forall\, k \in \mathbb{N}. \tag{8.13}$$

Note that

$$1 - e^{i\theta} \neq 0,$$

since $\theta \neq 2k\pi$ for all $k \in \mathbb{Z}$. Therefore we can use the Geometric Series (truncated) summation formula to get

$$\sum_{n=1}^{k} e^{i\theta n} = \sum_{n=1}^{k} e^{i\theta n}$$

$$= \frac{1 - e^{i(k+1)\theta}}{1 - e^{i\theta}} - 1$$

$$= e^{i\theta} \frac{1 - e^{ik\theta}}{1 - e^{i\theta}}$$

Thus

$$\left| \sum_{n=1}^{k} e^{i\theta n} \right| = \left| e^{i\theta} \frac{1 - e^{ik\theta}}{1 - e^{i\theta}} \right|$$

$$= \left| e^{i\theta} \right| \left| \frac{1 - e^{ik\theta}}{1 - e^{i\theta}} \right|$$

$$= \frac{|1 - e^{ik\theta}|}{|1 - e^{i\theta}|}$$

$$\leq \frac{|1| + |e^{ik\theta}|}{|1 - e^{i\theta}|}$$

$$= \frac{|1|}{|1 - e^{i\theta}|}$$

where we used the triangle inequality. Since the right hand side does not depend on k, we can set

$$M := \frac{|1|}{|1 - e^{i\theta}|},$$

so that (8.13) holds. Therefore

$$\sum_{n=1}^{\infty} \frac{e^{i\theta n}}{n}$$

converges by the Dirichlet Test. Recalling the Euler's Identity

$$e^{i\theta} = \cos(\theta) + i \sin(\theta),$$

we obtain that also the series

$$\sum_{n=1}^{\infty} \frac{\cos(n\theta)}{n}, \quad \sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n}$$

converge.

As a corollary of the Dirichlet Test we obtain the Alternate Convergence Test.

Theorem 8.56: Alternate Convergence Test

Let (q_n) be a sequence in $\mathbb{R}$ such that

- q_n is decreasing,
- $q_n \to 0$,
- $q_n \geq 0$ for all $n \in \mathbb{N}$.

The following series converges

$$\sum_{n=1}^{\infty} (-1)^n q_n$$

Proof

Define the sequence

$$c_n := (-1)^n.$$

Then

$$\sum_{n=1}^{k} c_n = \begin{cases} 0 & \text{if } k \text{ even} \\ -1 & \text{if } k \text{ odd} \end{cases}$$

Hence

$$\left| \sum_{n=1}^{k} c_n \right| \leq 1, \quad \forall\, k \in \mathbb{N}.$$

By the Dirichlet Test we have convergence of

$$\sum_{n=1}^{\infty} c_n q_n = \sum_{n=1}^{\infty} (-1)^n q_n.$$

Example 8.57

Prove that the below series converges conditionally

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}.$$

Solution. The series does not converge absolutely, since

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{1}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$$

diverges, being the Harmonic Series. We need to show convergence of

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}.$$

To this end, set

$$q_n := \frac{1}{n}.$$

Clearly $q_n \geq 0$, $q_n \to 0$ and q_n is decreasing. Hence the series converges by the Alternating Series Test.

8.5.6 Abel's Test

Another conditional convergence test is the Abel Test. This looks similar to the Dirichlet Test, however notice that the Abel Test only deals with real sequences.

Theorem 8.58: Abel's Test

Let (a_n) and (q_n) be sequences in $\mathbb{R}$. Suppose that

- q_n is monotone,
- q_n is bounded,
- The series below converges

$$\sum_{n=1}^{\infty} a_n.$$

Then the following series converges

$$\sum_{n=1}^{\infty} a_n q_n$$

The proof is similar to the one of the Dirichlet Test. We decided to omit it.

Example 8.59

Prove that the series below converges conditionally

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left(1 + \frac{1}{n}\right)^n .$$

Solution. Set

$$a_n := \frac{(-1)^n}{n}, \quad q_n := \left(1 + \frac{1}{n}\right)^n .$$

We have seen that q_n is monotone increasing and bounded. Moreover the series $\sum_{n=1}^{\infty} a_n$ converges by the Alternating Series Test, as seen in Example 8.57. Hence the series $\sum_{n=1}^{\infty} a_n q_n$ converges by the Abel Test.

However the series in question does not converge absolutely. Indeed,

$$\left| \frac{(-1)^n}{n} \left(1 + \frac{1}{n}\right)^n \right| = \frac{1}{n} q_n \geq \frac{1}{n} q_1 ,$$

since (q_n) is increasing. As the series

$$\sum_{n=1}^{\infty} \frac{1}{n} q_1$$

diverges, by the Comparison Test we conclude that also

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n} \left(1 + \frac{1}{n}\right)^n \right|$$

diverges, proving that the series in the example converges conditionally.

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